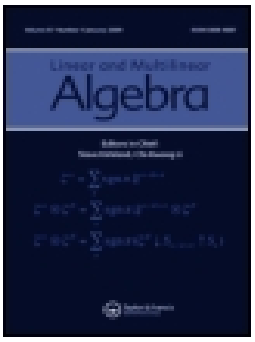

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Condition pseudospectral radius of bounded linear operators

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ABSTRACT
In this article, we consider the condition pseudospectrum of bounded linear operators on a Banach space. The condition pseudospectrum of normal matrices and Jordan blocks are characterized and condition pseudospectral radius of the classes are found. Sub-additivity and sub-multiplicativity of the condition pseudospectral radius for commuting pairs of bounded linear operators are proved. It is shown that the condition pseudospectral radius becomes a complete algebra norm in a commutative complex unital Banach algebra. Certain examples are given to illustrate the results. The results developed are also extended to a general setting.

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1. Introduction

Throughout this article, X denotes a complex Banach space and $BL(X)$ is the Banach algebra of all bounded linear operators on X. The condition pseudospectrum of an operator $A \in BL(X)$ is a generalization of the spectrum of A.

Definition 1.1: Let $A \in BL(X)$ and $0 < \epsilon < 1$. The ϵ-condition pseudospectrum of A is denoted by $\sigma_\epsilon(A)$ and is defined as

$$\sigma_\epsilon(A) := \sigma(A) \cup \left\{ z \in \mathbb{C} : \|zI - A\| \|(zI - A)^{-1}\| \geq \epsilon^{-1} \right\}.$$

Here $\sigma(A)$ is the spectrum of A and I is the identity operator in $BL(X)$. Condition pseudospectrum was introduced in the article [1]. In the same article, this generalized spectrum is named as the condition spectrum. Several properties of the spectrum are generalized to the condition pseudospectrum and compared to several other generalizations of spectrum, condition pseudospectrum is proved to be algebraically close to spectrum. Hence condition pseudospectrum is useful to study perturbation analysis of operators in $BL(X)$. Let $A \in BL(X)$ and $z, x, b \in X$. For $0 < \epsilon < 1$; $z \notin \sigma_\epsilon(A)$ guarantees a stable solution to the linear system $Ax - zx = b$. Thus, the condition pseudospectrum becomes a useful tool in

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the numerical solution of the system of linear equations and the numerical solution of differential equations. The condition pseudospectrum may also be used to study the norm behaviour of functions of operators in $BL(X)$. For more details and results on condition pseudospectrum, one may refer to [1–5].

Definition 1.2: Let $A \in BL(X)$ and $0 < \epsilon < 1$. The ϵ-condition pseudospectral radius of A is denoted by $r_\epsilon(A)$ and is defined as

$$r_\epsilon(A) := \sup\{|z| : z \in \sigma_\epsilon(A)\}.$$

The following properties of the condition pseudospectral radius are proved in [1].

Remark 1.3: Let $A \in BL(X)$ and $0 < \epsilon < 1$. Then

(1) $r(A) \leq r_\epsilon(A)$.
(2) $A = \alpha I$ if and only if $\sigma_\epsilon(A) = \{\alpha\}$.
(3) $r_\epsilon(A) \leq (1+\epsilon)/(1-\epsilon)\|A\|$.

The sub-additivity and sub-multiplicativity of the spectral radius and the pseudospectral radius are proved and are available in [6,7]. The purpose of the article is to prove the sub-additivity and sub-multiplicativity of the condition pseudospectral radius for a commuting pair of operators in $BL(X)$. The pseudospectrum is also a generalization of the spectrum and is used to study the norm behaviour of non-normal operators in $BL(X)$.

Definition 1.4: Let $A \in BL(X)$ and $\epsilon > 0$. The ϵ-pseudospectrum of A is denoted by $\Lambda_\epsilon(A)$ and is defined as

$$\Lambda_\epsilon(A) := \sigma(A) \cup \left\{z \in \mathbb{C} : \|(zI - A)^{-1}\| \geq \epsilon^{-1}\right\}.$$

Proposition 1.5: *Let $A \in BL(X)$ and $\epsilon > 0$. Further let $r(A)$ and $\rho_\epsilon(A)$ denote the spectral radius and pseudospectral radius of A, respectively. Then*

(1) *There exist an $A \neq 0$ such that $r(A) = 0$.*
(2) *If $A = 0$ then $\rho_\epsilon(A) = \epsilon$.*
(3) $r_\epsilon(A) = 0 \Leftrightarrow A = 0$.

Proof: (1) Consider $X = \mathbb{C}^2$ and $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.
(2) See [8].
(3) Follows from (2) of Remark 1.3. ∎

From (1) of Proposition 1.5, it follows that the spectral radius is not an algebra norm in $BL(X)$. From (2) of Proposition 1.5, we see that the pseudospectral radius also fails to become an algebra norm in $BL(X)$. The following is an outline of the article.

In section 2, the condition pseudospectrum of normal matrices and Jordan blocks are characterized. In Section 3, the sub-additivity and sub-multiplicativity of the condition pseudospectral radius for commuting pairs of operators of $BL(X)$ are proved. Certain examples are also given to illustrate the results. In Section 4, we prove that the condition pseudospectral radius and the operator norm on $BL(X)$ are equivalents. Hence, the condition spectral radius becomes a complete norm in a commutative complex unital Banach algebra. In Section 5, the results proved in the previous sections are extended to non-commutative pairs of operators in $BL(X)$ and almost commutative pairs of matrices.

2. Condition pseudospectrum of normal matrices and Jordan blocks

In this section, we characterize the condition pseudospectrum and condition pseudospectral radius of normal matrices and Jordan blocks. For the normal matrix, we take $\|\cdot\|_2$ and for the Jordan block we take $\|\cdot\|_1$ or $\|\cdot\|_\infty$. For $z_0 \in \mathbb{C}$ and $r > 0$ define

$$D(z_0, r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

Theorem 2.1: *Let $A \in \mathbb{C}^{N\times N}$ be a normal matrix with distinct eigenvalues $\{\lambda_1, \ldots, \lambda_k\}$ and $0 < \epsilon < 1$. Then*

$$\sigma_\epsilon(A) = \bigcup_{p,q=1}^{k} D\left(\frac{\lambda_p - \epsilon^2\lambda_q}{1-\epsilon^2}, \frac{\epsilon\,|\lambda_p - \lambda_q|}{1-\epsilon^2}\right).$$

Proof: We have $A = QDQ^*$ for some Q unitary and D diagonal with entries as eigenvalues of A. Since Q is unitary $\|zI - A\|\,\|(zI - A)^{-1}\| = \|zI - D\|\,\|(zI - D)^{-1}\|$. We also have

$$\|zI - D\| = \max_{\lambda_p \in \sigma(A)} |z - \lambda_p| \quad \text{and} \quad \|(zI - D)^{-1}\| = \frac{1}{\min\limits_{\lambda_q \in \sigma(A)} |z - \lambda_q|}.$$

Thus for $0 < \epsilon < 1$

$$\begin{aligned}
\sigma_\epsilon(A) &= \left\{ z \in \mathbb{C} : \frac{\max\limits_{\lambda_p \in \sigma(A)} |z-\lambda_p|}{\min\limits_{\lambda_q \in \sigma(A)} |z-\lambda_q|} \geq \frac{1}{\epsilon} \right\} \\
&= \bigcup_{\lambda_p,\lambda_q \in \sigma(A)} \left\{ z \in \mathbb{C} : \frac{|z-\lambda_p|}{|z-\lambda_q|} \geq \frac{1}{\epsilon} \right\} \\
&= \bigcup_{\lambda_p,\lambda_q \in \sigma(A)} \left\{ z \in \mathbb{C} : |z-\lambda_p| \leq \epsilon |z-\lambda_q| \right\}.
\end{aligned}$$

Denote $z = x + iy$, $\lambda_p = r_p + i\,s_p$ where $r_p, s_p \in \mathbb{R}$. Then $|z - \lambda_p| \leq \epsilon|z - \lambda_q|$ becomes

$$\left(x - \frac{r_p - \epsilon^2 r_q}{1-\epsilon^2}\right)^2 + \left(y - \frac{s_p - \epsilon^2 s_q}{1-\epsilon^2}\right)^2 \leq \frac{\epsilon^2[(r_p - r_q)^2 + (s_p - s_q)^2]}{(1-\epsilon^2)^2}$$

and $\sigma_\epsilon(A)$ becomes the union of disks with centres $\left(\frac{r_p - \epsilon^2 r_q}{1-\epsilon^2}, \frac{s_p - \epsilon^2 s_q}{1-\epsilon^2}\right) = \frac{\lambda_p - \epsilon^2\lambda_q}{1-\epsilon^2}$ and the corresponding radii $\frac{\epsilon\sqrt{(r_p - r_q)^2 + (s_p - s_q)^2}}{1-\epsilon^2} = \frac{\epsilon\,|\lambda_p - \lambda_q|}{1-\epsilon^2}$. ■

Remark 2.2: Let $A \in \mathbb{C}^{N\times N}$ be a normal matrix with distinct eigenvalues $\{\lambda_1,\dots,\lambda_k\}$ and $0<\epsilon<1$. Then

$$r_\epsilon(A) = \frac{1}{1-\epsilon^2}\max\{|\lambda_p - \epsilon^2\lambda_q| + \epsilon|\lambda_p-\lambda_q| : \lambda_p,\lambda_q \in \sigma(A)\}.$$

Example 2.3: Consider the 3×3 Hilbert matrix $A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\end{bmatrix}$. We have $\sigma(A) = \{0.003, 0.122, 1.408\}$. Denote $\lambda_1 = 0.003, \lambda_2 = 0.122, \lambda_3 = 1.408$. Then for $\epsilon = 0.25$ we have

$\lvert\lambda_1-\epsilon^2\lambda_2\rvert = 0.008$	$\epsilon\lvert\lambda_1-\lambda_2\rvert = 0.298$	$\lvert\lambda_1-\epsilon^2\lambda_2\rvert+\epsilon\lvert\lambda_1-\lambda_2\rvert = 0.306$
$\lvert\lambda_1-\epsilon^2\lambda_3\rvert = 0.085$	$\epsilon\lvert\lambda_1-\lambda_3\rvert = 0.351$	$\lvert\lambda_1-\epsilon^2\lambda_3\rvert+\epsilon\lvert\lambda_1-\lambda_3\rvert = 0.436$
$\lvert\lambda_2-\epsilon^2\lambda_1\rvert = 0.122$	$\epsilon\lvert\lambda_2-\lambda_1\rvert = 0.298$	$\lvert\lambda_2-\epsilon^2\lambda_1\rvert+\epsilon\lvert\lambda_2-\lambda_1\rvert = 0.420$
$\lvert\lambda_2-\epsilon^2\lambda_3\rvert = 0.034$	$\epsilon\lvert\lambda_2-\lambda_3\rvert = 0.322$	$\lvert\lambda_2-\epsilon^2\lambda_3\rvert+\epsilon\lvert\lambda_2-\lambda_3\rvert = 0.356$
$\lvert\lambda_3-\epsilon^2\lambda_1\rvert = 1.408$	$\epsilon\lvert\lambda_3-\lambda_1\rvert = 0.351$	$\lvert\lambda_3-\epsilon^2\lambda_1\rvert+\epsilon\lvert\lambda_3-\lambda_1\rvert = 1.759$
$\lvert\lambda_3-\epsilon^2\lambda_2\rvert = 1.400$	$\epsilon\lvert\lambda_3-\lambda_2\rvert = 0.322$	$\lvert\lambda_3-\epsilon^2\lambda_2\rvert+\epsilon\lvert\lambda_3-\lambda_2\rvert = 1.722$

Thus $\sigma_{0.25}(A)$ is the union of the disks $D(0.009, 0.318)$, $D(0.091, 0.374)$, $D(0.130, 0.318)$, $D(0.036, 0.343)$, $D(1.502, 0.374)$ and $D(1.493, 0.343)$. We also have $r_{0.25}(A) = 1.876$.

Theorem 2.4: *Let* $A = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \lambda \end{bmatrix}$ *be an* $N\times N$ *Jordan block,* $\|\cdot\| = \|\cdot\|_1$ *or* $\|\cdot\|_\infty$ *and* $0<\epsilon<1$. *Then* $\sigma_\epsilon(A) = \lambda + \sigma_\epsilon(A_0)$ *where* $A_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 0 \end{bmatrix}$ *and* $\sigma_\epsilon(A_0) = \{z \in \mathbb{C} : \dfrac{|z|^N}{(1+|z|)(1+|z|+\cdots+|z|^{N-1})} \le \epsilon\}$.

Proof: We have $A = \lambda I + A_0$ and $\sigma_\epsilon(A) = \lambda + \sigma_\epsilon(A_0)$ [1]. For $z \in \mathbb{C}$

$$\|zI - A_0\| = 1 + |z| \text{and} \|(zI-A_0)^{-1}\| = \frac{1+|z|+\cdots+|z|^{N-1}}{|z|^N}.$$

Then

$$\sigma_\epsilon(A_0) = \left\{ z\in\mathbb{C} : \frac{(1+|z|)(1+|z|+\cdots+|z|^{N-1})}{|z|^N} \ge \epsilon^{-1}\right\}$$

and $\sigma_\epsilon(A) = \lambda + \sigma_\epsilon(A_0)$. ■

Proposition 2.5: *Let A_0 be the Jordan block defined in Theorem 2.4. Then*

(1) $r_\epsilon(A_0) = 1$ *if and only if* $\epsilon = \dfrac{1}{2N}$.

(2) $r_\epsilon(A_0) < 1$ *if and only if* $(1-\epsilon) r_\epsilon(A_0)^N - \dfrac{2\epsilon\, r_\epsilon(A_0)[1 - r_\epsilon(A_0)^N]}{1 - r_\epsilon(A_0)} = \epsilon$.

(3) $r_\epsilon(A_0) > 1$ *if and only if* $(1-\epsilon) r_\epsilon(A_0)^N - \dfrac{2\epsilon\, r_\epsilon(A_0)[r_\epsilon(A_0)^N - 1]}{r_\epsilon(A_0) - 1} = \epsilon$.

(4) $\sigma_\epsilon(A) = D(\lambda, r_\epsilon(A_0))$.

Proof: Let $0 < \epsilon < 1$ and $|z| = r_\epsilon(A_0)$ then $\dfrac{(1+|z|)(1+|z|+\cdots+|z|^{N-1})}{|z|^N} = \dfrac{1}{\epsilon}$ and (1), (2) and (3) follows. Define $\pi_\epsilon(A) := \{z \in \sigma_\epsilon(A) : |z| = r_\epsilon(A)\}$. From Theorem 2.4, if $z \in \sigma_\epsilon(A)$ then $|z|e^{i\theta} \in \sigma_\epsilon(A)$ for every $-\pi < \theta \leq \pi$. It is also true that if $|z| = r_\epsilon(A_0)$ then $\pi_\epsilon(A_0) = |z|e^{i\theta}$ for every $-\pi < \theta \leq \pi$. Thus $\sigma_\epsilon(A_0) = D(0, r_\epsilon(A_0))$ and $\sigma_\epsilon(A) = D(\lambda, r_\epsilon(A_0))$. ■

Let A be the 5×5 Jordan block. The following table gives $r_\epsilon(A)$ for various values of λ and ϵ using the python programme.

ϵ	$r_\epsilon(A_0)$	λ	$r_\epsilon(A)$	$\sigma_\epsilon(A)$
0.01	0.5000	$1+i$	1.9142	$D(1+i, 0.5)$
0.02	0.6000	$2-i$	2.8661	$D(2-i, 0.6)$
0.025	0.6300	$3+2i$	4.2356	$D(3+2i, 0.63)$
0.3	1.8000	i	2.8000	$D(i, 1.8)$
0.4	2.2999	$-i$	3.3999	$D(-i, 2.3)$
0.45	2.6199	$3-2i$	6.2255	$D(3-2i, 2.62)$
0.5	2.9899	$2+i$	5.2260	$D(2+i, 2.99)$
0.75	6.9999	$1-i$	8.4141	$D(1-i, 7)$

3. Sub-additivity and sub-multiplicativity for commuting pairs of operators

Let $\mathcal{I} := \{zI : z \in \mathbb{C}\}$. The following lemma and its proof are modifications of Theorem 1 in [6, p.18] and of Lemma 3.1 in [7] and their proofs.

Lemma 3.1: *Let Γ be a bounded semi-group of $BL(X)$ under multiplication (or composition) and $I \in \Gamma$. Then there exists a function $p : BL(X) \to \mathbb{R}^+$ satisfying the following conditions.*

(1) $r_\epsilon(A) \leq p(A)$ *for all* $A \in BL(X)$ *and* $0 < \epsilon < 1$.
(2) $p(S) \leq 1$ *for all* $S \in \Gamma$.
(3) $p(A+B) \leq p(A) + p(B)$ *for all* $A, B, A+B \notin \mathcal{I}$.
(4) $p(AB) \leq p(A)\,p(B)$ *for all* $A, B \in BL(X)$.

Proof: For $0 < \epsilon < 1$ define $q : BL(X) \to \mathbb{R}^+$ by

$$q(A) = \begin{cases} \sup\left\{\dfrac{1+\epsilon}{1-\epsilon}\|SA\| : S \in \Gamma\right\} & A \notin \mathcal{I}; \\ |\alpha| & A = \alpha I. \end{cases}$$

The following properties of q can be easily verified:

(a) If $A \in \mathcal{I}$, then $r_\epsilon(A) = q(A)$.
(b) If $A \in BL(X) \setminus \mathcal{I}$, then $r_\epsilon(A) \leq q(A)$.
(c) $q(\alpha A) = |\alpha| q(A)$ for all $A \in BL(X)$ and $\alpha \in \mathbb{C}$.
(d) $q(A+B) \leq q(A) + q(B)$ for all $A, B, A+B \notin \mathcal{I}$.

(e) $q(AB) \leq q(A)\, q(B)$ for all $A, B \in BL(X)$.

Define $p : BL(X) \to \mathbb{R}^+$ as

$$p(A) = \sup\{q(AX) : X \in \Gamma, q(X) \leq 1\}.$$

We claim that $p(A) = q(A)$ for all $A \in BL(X)$.
Since $I \in \Gamma$ and $q(I) = 1$, $q(A) \leq p(A)$ for all $A \in BL(X)$. Also

$$p(A) \leq \sup\{q(A)\, q(X) : X \in \Gamma, q(X) \leq 1\} \leq q(A).$$

This proves the claim. Now we are ready to prove the four conditions of p stated. Since $p(A) = q(A)$ for all $A \in BL(X)$,

(1) follows from (a) and (b).
(2) Recall that $p(I) = q(I) = 1$. For $S \in \Gamma \setminus \{I\}$,

$$\begin{aligned} q(SX) &= \sup\left\{\frac{1+\epsilon}{1-\epsilon}\|S'SX\| : S' \in \Gamma\right\} \\ &\leq \sup\left\{\frac{1+\epsilon}{1-\epsilon}\|S_1 x\| : S_1 \in \Gamma\right\} \text{ (since } \Gamma \text{ is a bounded semi} - \text{group)} \\ &= q(X). \end{aligned}$$

From the definition of $p(S)$, it follows that

$$p(S) = \sup_{q(X)\leq 1} q(SX) \leq \sup_{q(X)\leq 1} q(X) = 1.$$

(3) follows from (d).
(4) follows from (e). ■

Now we are ready to prove the sub-additivity and sub-multiplicativity of the condition pseudospectral radius for a commuting pair of operators in $BL(X)$. The following theorems are suitable modifications of Theorem 3.2 and 3.3 of [7].

Theorem 3.2: *Let $A, B \in BL(X)$ and $AB = BA$. Then $r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B)$ for all* $0 < \epsilon < 1$.

Proof: Let $0 < \epsilon < 1$. We consider all the possible forms of A and B.
Case 1: If $A, B \in \mathcal{I}$, then $A + B \in \mathcal{I}$ and

$$r_\epsilon(A + B) = r(A + B) \leq r(A) + r(B) = r_\epsilon(A) + r_\epsilon(B).$$

Case 2: If $A \in \mathcal{I}$ or $B \in \mathcal{I}$. Further assume that $A = \alpha I$ for some $\alpha \in \mathbb{C}$. Then, [1],

$$\sigma_\epsilon(A + B) = \sigma_\epsilon(\alpha I + B) = \alpha + \sigma_\epsilon(B) = \sigma_\epsilon(A) + \sigma_\epsilon(B)$$

and

$$r_\epsilon(A + B) = r_\epsilon(A) + r_\epsilon(B).$$

Case 3: If $A, B \notin \mathcal{I}$ and $A + B \in \mathcal{I}$. Then

$$r_\epsilon(A+B) = r(A+B) \leq r(A) + r(B) \leq r_\epsilon(A) + r_\epsilon(B).$$

Case 4: If $A, B, A + B \notin \mathcal{I}$. For $\delta > 0$ define

$$U = \frac{A}{r_\epsilon(A)+\delta} \quad \text{and} \quad V = \frac{B}{r_\epsilon(B)+\delta}.$$

Note that both $U, V \notin \mathcal{I}$, $UV = VU$, $r(U) < 1$ and $r(V) < 1$. Thus the set $\Gamma := \{U^i V^j : i,j \geq 0\}$ becomes a bounded semi-group under multiplication [7,9]. From Lemma 3.1, there exists $p : BL(X) \to \mathbb{R}^+$ such that

$$p\left(\frac{A}{r_\epsilon(A)+\delta}\right) = \frac{p(A)}{r_\epsilon(A)+\delta} \leq 1,$$
$$p\left(\frac{B}{r_\epsilon(B)+\delta}\right) = \frac{p(B)}{r_\epsilon(B)+\delta} \leq 1.$$

Thus $p(A) \leq r_\epsilon(A) + \delta$ and $p(B) \leq r_\epsilon(B) + \delta$. Together with other properties of p,

$$r_\epsilon(A+B) \leq p(A+B) \leq p(A) + p(B) \leq r_\epsilon(A) + r_\epsilon(B) + 2\delta.$$

Since $\delta > 0$ is arbitrary, we have $r_\epsilon(A+B) \leq r_\epsilon(A) + r_\epsilon(B)$. ■

Theorem 3.3: *Let $A, B \in BL(X)$ and $AB = BA$. Then $r_\epsilon(AB) \leq r_\epsilon(A)\ r_\epsilon(B)$ for all $0 < \epsilon < 1$.*

Proof: Let $0 < \epsilon < 1$. We consider all the possible forms of A and B.

Case 1: If $A, B \in \mathcal{I}$. Then $AB \in \mathcal{I}$ and

$$r_\epsilon(AB) = r(AB) \leq r(A)\ r(B) = r_\epsilon(A)\ r_\epsilon(B).$$

Case 2: If $A \in \mathcal{I}$ or $B \in \mathcal{I}$. Further assume that $A = \alpha I$ for some $\alpha \in \mathbb{C}$. Then, [1],

$$\sigma_\epsilon(AB) = \sigma_\epsilon(\alpha B) = \alpha\ \sigma_\epsilon(B) = \sigma_\epsilon(A)\ \sigma_\epsilon(B)$$

and

$$r_\epsilon(AB) = r_\epsilon(A)\ r_\epsilon(B).$$

Case 3: If $A, B \notin \mathcal{I}$ and $AB \in \mathcal{I}$. Then

$$r_\epsilon(AB) = r(AB) \leq r(A)\ r(B) \leq r_\epsilon(A)\ r_\epsilon(B).$$

Case 4: If $A, B, AB \notin \mathcal{I}$. For $\delta > 0$ define

$$U = \frac{A}{r_\epsilon(A)+\delta} \quad \text{and} \quad V = \frac{B}{r_\epsilon(B)+\delta}.$$

Note that both $U, V \notin \mathcal{I}$, $UV = VU$, $r(U) < 1$ and $r(V) < 1$. Then the set $\{U^i V^j : i,j \geq 0\}$ is a bounded semi-group under multiplication [7,9]. From Lemma 3.1, there exists

$p : BL(X) \to \mathbb{R}^+$ such that

$$p(U) = p\left(\frac{A}{r_\epsilon(A)+\delta}\right) \le 1,$$
$$p(V) = p\left(\frac{B}{r_\epsilon(B)+\delta}\right) \le 1.$$

Thus $p(A) \le r_\epsilon(A) + \delta$ and $p(B) \le r_\epsilon(B) + \delta$. Together with the other properties of p,

$$r_\epsilon(AB) \le p(AB) \le p(A)\,p(B) \le (r_\epsilon(A)+\delta)(r_\epsilon(B)+\delta).$$

Since $\delta > 0$ is arbitrary, we have $r_\epsilon(AB) \le r_\epsilon(A)\,r_\epsilon(B)$. ■

In the following, we give a noncommutaive pair of matrices where the sub-additivity and sub-multiplicativity of the condition pseudospectral radius fail.

Example 3.4: Consider $BL(\mathbb{C}^2)$. Let $t > 0$ and $A = \left[\begin{smallmatrix} 0 & t^2 \\ 1 & 0 \end{smallmatrix}\right]$. Then $AA^* \ne A^*A$. We also have

$$\sigma(A) = \sigma(A^*) = \{t, -t\}, \quad \sigma(AA^*) = \{1, t^4\}, \quad \sigma(A + A^*) = \{1 + t^2, -1 - t^2\}.$$

Hence

$$r(A) = r(A^*) = t, r(AA^*) = \max\{1, t^4\} \quad \text{and} \quad r(A + A^*) = 1 + t^2.$$

Since $\bigcap\limits_{0<\epsilon<1} \sigma_\epsilon(A) = \sigma(A)$ [1], we have $\lim\limits_{\epsilon\to 0} r_\epsilon(A) = r(A)$. Thus for sufficiently small ϵ, t

$$r_\epsilon(A + A^*) > r_\epsilon(A) + r_\epsilon(A^*),$$
$$r_\epsilon(AA^*) > r_\epsilon(A)\,r_\epsilon(A^*).$$

The following examples give two noncommutative pairs of matrices where the sub-additivity and sub-multiplicativity of the condition pseudospectral radius hold. Thus, the commutativity of the operators is not necessary for the sub-additivity and sub-multiplicativity of the condition pseudospectral radius.

Example 3.5: Consider $BL(\mathbb{C}^2)$ with $\|\cdot\|_1$. Let $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

Then $AB \ne BA, A + B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and $r_\epsilon(A) \ge 1$ for all $0 < \epsilon < 1$. For $\lambda \in \mathbb{C}$, we have

$$\|\lambda I - B\|_1 = 1 + |\lambda| \text{and} \|(\lambda I - B)^{-1}\|_1 = \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}.$$

Then

$$\sigma_\epsilon(B) = \left\{\lambda \in \mathbb{C} : \|\lambda I - B\|_1 \|(\lambda I - B)^{-1}\|_1 \ge \epsilon^{-1}\right\}.$$
$$= \left\{\lambda \in \mathbb{C} : |\lambda| \le \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}\right\}.$$

Thus $r_\epsilon(B) = \dfrac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}$ and $r_\epsilon(A) + r_\epsilon(B) \ge 1 + \dfrac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}$. From Remark 1.3, we also have

$$r_\epsilon(A + B) \le \frac{1+\epsilon}{1-\epsilon}\|A + B\|_1 = \frac{1+\epsilon}{1-\epsilon} = 1 + \frac{2\epsilon}{1-\epsilon}.$$

Then for $\epsilon < 1/9$ we have $r_\epsilon(A + B) \le r_\epsilon(A) + r_\epsilon(B)$.

Example 3.6: Consider $BL(\mathbb{C}^2)$. Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AB = 0$ and $BA \neq 0$. From Remark 1.3, we have $r_\epsilon(AB) = 0$ for all $0 < \epsilon < 1$. Since $A \neq 0 \neq B$ we have $r_\epsilon(A) > 0$ and $r_\epsilon(B) > 0$ for all $0 < \epsilon < 1$ [1]. Thus

$$r_\epsilon(AB) \leq r_\epsilon(A)\, r_\epsilon(B) \quad \text{for all } 0 < \epsilon < 1.$$

In the following theorem, we show that the condition pseudospectral radius is equivalent to the operator norm.

Theorem 3.7: *Let $A \in BL(X)$ and $0 < \epsilon < 1$. Then*

$$\|A\| \leq (2 + 3\epsilon^{-1})\, r_\epsilon(A) \leq \frac{1+\epsilon}{1-\epsilon}(2 + 3\epsilon^{-1})\, \|A\|.$$

Proof: Define $\delta := r_\epsilon(A)$. Then $2\delta \notin \sigma_\epsilon(A)$ and

$$\|2\delta I - A\|\,\|(2\delta I - A)^{-1}\| < \epsilon^{-1}. \tag{1}$$

Since $\sigma(A) \subseteq \sigma_\epsilon(A)$, we have $D(0, \delta) := \{z \in \mathbb{C} : |z| \leq \delta\}$ contains an element of $\sigma(A)$ and $\operatorname{dist}(2\delta, \sigma(A)) \leq 3\delta$. On the other hand from elementary perturbation theory, we have

$$\operatorname{dist}(2\delta, \sigma(A)) \geq \|(2\delta - A)^{-1}\|^{-1}. \tag{2}$$

Together with (1) and (2), we have

$$\|A\| - 2\delta \leq \|2\delta I - A\| < \frac{\|(2\delta I - A)^{-1}\|^{-1}}{\epsilon} \leq \frac{\operatorname{dist}(2\delta, \sigma(A))}{\epsilon} \leq \frac{3\delta}{\epsilon}.$$

This gives

$$\|A\| \leq (2 + 3\epsilon^{-1}) r_\epsilon(A) \leq \frac{1+\epsilon}{1-\epsilon}(2 + 3\epsilon^{-1})\|A\|.$$

The last inequality follows from (3) of Remark 1.3. ■

4. Condition pseudospectral radius: a complete norm in a commutative Banach algebra

In section 3, we have proved the sub-additivity and sub-multiplicativity of the condition pseudospectral radius for commuting pairs of operators in $BL(X)$. The results developed are also true for commuting elements in a complex unital Banach algebra. The proof of the result in this general setting follows exactly as that of $BL(X)$. In the present section, we show that the condition pseudospectral radius becomes a complete algebra norm in a commutative complex unital Banach algebra.

Theorem 4.1: *Let $\mathcal{A}$ be a complex unital Banach algebra and $0 < \epsilon < 1$. Then*

(1) $r_\epsilon(a) = 0 \Leftrightarrow a = 0$.
(2) $r_\epsilon(\alpha a) = |\alpha|\, r_\epsilon(a) \quad \forall a \in \mathcal{A}, \alpha \in \mathbb{C}$.

(3) $r_\epsilon(a+b) \le r_\epsilon(a) + r_\epsilon(b) \quad \forall a, b \in \mathcal{A}$ and $ab = ba$.
(4) $r_\epsilon(ab) \le r_\epsilon(a) r_\epsilon(b) \quad \forall a, b \in \mathcal{A}$ and $ab = ba$.

Proof: Let $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$.

(1) Since $\sigma_\epsilon(a) = \{0\}$ if and only if $a = 0$ [1],

$$r_\epsilon(a) = 0 \text{ if and only if } a = 0.$$

(2) Since $\sigma_\epsilon(\alpha a) = \alpha\, \sigma_\epsilon(a)$ [1], we have $r_\epsilon(\alpha a) = |\alpha|\, r_\epsilon(a)$.
(3) Follows similarly as Theorem 3.2.
(4) Follows similarly as Theorem 3.3. ■

Corollary 4.2: *Let $(\mathcal{A}, \|\cdot\|)$ be a commutative complex unital Banach algebra and $0 < \epsilon < 1$. Then the condition pseudospectral radius $r_\epsilon : \mathcal{A} \to \mathbb{R}^+$ is a complete algebra norm on $\mathcal{A}$.*

Proof: From Theorem 4.1, r_ϵ is an algebra norm on $\mathcal{A}$ for every $0 < \epsilon < 1$. We also have r_ϵ and $\|\cdot\|$ are equivalent (the proof is similar to Theorem 3.7). Thus r_ϵ becomes a complete algebra norm for every $0 < \epsilon < 1$. ■

Remark 4.3: Let X be a compact Hausdorff space and

$$C(X) := \{f : f : X \to \mathbb{C} \text{ and } f \text{ is continuous}\}.$$

For $f \in C(X)$ define $\|f\| := \sup\{|f(x)| : x \in \mathbb{C}\}$. Then $C(X)$ is a commutative complex unital Banach algebra and $\sigma(f) = \{f(x) : x \in X\}$. For $z \notin \sigma(f)$,

$$\|ze - f\| = \max\{|z - f(x)| : x \in X\},$$
$$\|(ze - f)^{-1}\| = \max\left\{\frac{1}{|z - f(x)|} : x \in X\right\}.$$

Here $e : X \to \mathbb{C}$ and $e(x) = 1$ for every $x \in X$. For $0 < \epsilon < 1$ we have

$$\sigma_\epsilon(f) = \left\{z \in \mathbb{C} : \frac{\max\{|z - f(x)| : x \in X\}}{\min\{|z - f(x)| : x \in X\}} \ge \frac{1}{\epsilon}\right\}.$$

Example 4.4: Consider $C([0,1])$ and define $f \in C([0,1])$ as

$$f(t) = t \text{ for every } t \in [0,1].$$

Then $\sigma(f) = [0,1]$. For $0 < \epsilon < 1$ we have

$$\sigma_\epsilon(f) = \left\{z = x + iy \in \mathbb{C} : \frac{\max\{|z - t| : t \in [0,1]\}}{\min\{|z - t| : t \in [0,1]\}} \ge \frac{1}{\epsilon}\right\}.$$

(1) Let $x = \frac{1}{2}$. Then

$$\sigma_\epsilon(f) = \left\{\frac{1}{2} + iy : \frac{\sqrt{\frac{1}{4} + y^2}}{|y|} \geq \frac{1}{\epsilon}\right\}$$
$$= \left\{\frac{1}{2} + iy : |y| \leq \frac{\epsilon}{2\sqrt{1-\epsilon^2}}\right\}.$$

(2) $x < \frac{1}{2}$. Then

$$\sigma_\epsilon(f) = \left\{x + iy : \frac{\sqrt{(x-1)^2 + y^2}}{|y|} \geq \frac{1}{\epsilon}\right\}$$
$$= \left\{x + iy : |y| \leq \frac{\epsilon}{\sqrt{1-\epsilon^2}}(x-1)\right\}.$$

(3) $x > \frac{1}{2}$. Then

$$\sigma_\epsilon(f) = \left\{x + iy : \frac{\sqrt{x^2 + y^2}}{|y|} \geq \frac{1}{\epsilon}\right\}$$
$$= \left\{x + iy : |y| \leq \frac{\epsilon}{\sqrt{1-\epsilon^2}}x\right\}.$$

Thus $\sigma_\epsilon(f)$ is the portion joined by the points $(0,0)$, $\left(\frac{1}{2}, \frac{\epsilon}{2\sqrt{1-\epsilon^2}}\right)$, $(1,0)$ and $\left(\frac{1}{2}, -\frac{\epsilon}{2\sqrt{1-\epsilon^2}}\right)$. Hence $r_\epsilon(f) = \max\left\{1, \frac{1}{2\sqrt{1-\epsilon^2}}\right\}$.

5. For non-commuting pairs of operators in *BL*(*X*)

In this section, we consider a non-commuting pair of operators in $BL(X)$ and prove results similar to Theorem 3.2 and Theorem 3.3. We are seeking the sub-additivity and sub-multiplicativity of the condition pseudospectral radius for a non-commuting pair of operators in $BL(X)$. Let $A, B \in BL(X)$ and $AB \neq BA$. In this case, we need to look for a commuting pair of operators $A_1, B_1 \in BL(X)$ such that A_1, B_1 are, respectively, close to A, B. Define

$$\rho := \min_{A_1B_1=B_1A_1} \max\{\|A - A_1\|, \|B - B_1\|\}.$$

Let $0 < \epsilon < 1$. Since the map $A \mapsto \sigma_\epsilon(A)$ is upper semicontinuous [1], $A \mapsto r_\epsilon(A)$ is a continuous map and

$$|r_\epsilon(A) - r_\epsilon(A_1)| \leq f(A, A_1, \epsilon, \rho) \tag{3}$$
$$|r_\epsilon(B) - r_\epsilon(B_1)| \leq g(B, B_1, \epsilon, \rho) \tag{4}$$
$$|r_\epsilon(A+B) - r_\epsilon(A_1+B_1)| \leq h(A, B, A_1, B_1, \epsilon, \rho) \tag{5}$$
$$|r_\epsilon(AB) - r_\epsilon(A_1B_1)| \leq k(A, B, A_1, B_1, \epsilon, \rho) \tag{6}$$

for some continuous functions f, g, h and k. For $\delta > 0$ define

$$U := \frac{A_1}{r_\epsilon(A_1) + \delta} \text{ and } V := \frac{B_1}{r_\epsilon(B_1) + \delta}.$$

Then $r(U) < 1, r(V) < 1$ and $UV = VU$. Thus $\Gamma := \{U^i V^j : i, j \geq 0\}$ becomes a bounded semi-group under multiplication. From Lemma 3.1, there exists a function $p : BL(X) \to \mathbb{R}^+$ satisfying all four conditions stated in the lemma. In particular,

$$p\left(\frac{A_1}{r_\epsilon(A_1) + \delta}\right) \leq 1 \quad \text{and} \quad p\left(\frac{B_1}{r_\epsilon(B_1) + \delta}\right) \leq 1.$$

Also

$$p(A_1) \leq r_\epsilon(A_1) + \delta \quad \text{and} \quad p(B_1) \leq r_\epsilon(B_1) + \delta.$$

Thus,

$$r_\epsilon(A + B) \leq r_\epsilon(A_1 + B_1) + h \ \leq \ p(A_1 + B_1) + h \tag{7}$$
$$\leq p(A_1) + p(B_1) + h \tag{8}$$
$$\leq r_\epsilon(A_1) + \delta + r_\epsilon(B_1) + \delta + h \tag{9}$$
$$\leq r_\epsilon(A) + r_\epsilon(B) + f + g + h + 2\delta. \tag{10}$$

Since $\delta > 0$ is as an arbitrary positive value by letting $\delta \to 0$, we have

$$r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B) + f + g + h.$$

Whenever A, B commute, choose $f = g = h = 0$ and we have Theorem 3.2. We also have

$$r_\epsilon(AB) \leq r_\epsilon(A_1 B_1) + k \ \leq \ p(A_1 B_1) + k \tag{11}$$
$$\leq p(A_1)p(B_1) + k \tag{12}$$
$$\leq (r_\epsilon(A_1) + \delta)(r_\epsilon(B_1) + \delta) + k \tag{13}$$
$$\leq (r_\epsilon(A) + f + \delta)(r_\epsilon(B) + g + \delta) + k. \tag{14}$$

By letting $\delta \to 0$ we obtain

$$r_\epsilon(AB) \leq (r_\epsilon(A) + f)(r_\epsilon(B) + g) + k$$

Whenever A, B commute, choose $f = g = k = 0$ and we have Theorem 3.3.

The following proposition follows from Theorem 3.7.

Proposition 5.1: *Let $A, B \in BL(X)$ and $0 < \epsilon < 1$. Then*

(1) $r_\epsilon(A + B) \leq \dfrac{1 + \epsilon}{1 - \epsilon}(2 + 3\epsilon^{-1})[r_\epsilon(A) + r_\epsilon(B)]$.

(2) $r_\epsilon(AB) \leq \dfrac{1 + \epsilon}{1 - \epsilon}(2 + 3\epsilon^{-1})^2[r_\epsilon(A)\, r_\epsilon(B)]$

Note 5.2: In Proposition 5.1 define $f(\epsilon) = \frac{1+\epsilon}{1-\epsilon}(2+3\epsilon^{-1})$ and $g(\epsilon) = \frac{1+\epsilon}{1-\epsilon}(2+3\epsilon^{-1})^2$. Then

(1) $f\left(\frac{\sqrt{30}-3}{7}\right) \leq f(\epsilon)$ for every $0 < \epsilon < 1$.
(2) $g(0.5306) \leq f(\epsilon)$ for every $0 < \epsilon < 1$.

5.1. Almost commuting finite-dimensional operators

Definition 5.3: Let $A, B \in BL(X)$ and $\delta > 0$. Then A and B are said to be δ-almost commutative if $\|AB - BA\| \leq \delta$. If $\delta > 0$ is sufficiently small then A, B are called almost commutative.

The following results for almost commuting matrices are available in the literature.

(1) Let $A, B \in BL(\mathbb{C}^N)$ with $A = A^*$ and $\|AB - BA\| \leq \frac{2\delta^2}{N-1}$ for some $\delta \geq 0$. Then there exist $A_1, B_1 \in \mathbb{C}^{N\times N}$ with $A_1^* = A_1$ such that $A_1B_1 = B_1A_1$, $\|A - A_1\| \leq \delta$ and $\|B - B_1\| \leq \delta$ [10].
(2) There exist $A, B \in BL(\mathbb{C}^N)$ such that $\|AB - BA\| \leq \delta$ for some $\delta > 0$ and A, B may not be near to any commuting pairs [11].
(3) Let $A, B \in BL(\mathbb{C}^N)$ such that $\|A\| \leq 1, \|B\| \leq 1$ and $\|AB - BA\| \leq \delta$ for some $\delta \geq 0$. Using non-standard analysis, the authors proved in [12] that there exists a commuting pair A', B' with $\|A'\| \leq 1, \|B'\| \leq 1$ such that $\|A - A'\| \leq f_N(\delta)$ and $\|B - B'\| \leq f_N(\delta)$. The constant $f_N(\delta)$ is dependent on the pair A, B and the order of the matrices N, such that $f_N(\delta) \to 0$ as $\delta \to 0$.

Thus (1), (2) and (3) together show that finding a quantity independent of the order of the matrix and depending only on the constant δ is not possible. If A is self-adjoint, then it is possible to find a constant which is independent of the order of the matrices. The following lemma is the condition pseudospectral version of Lemma 4.1 in [7] (the pseudospectral version).

Lemma 5.4: *Let $A \in BL(\mathbb{C}^N)$ and $0 < \epsilon < 1$. Further assume that there exists $z \in \sigma_\epsilon(A)$ such that $|z| = r_\epsilon(A)$. Then for any $c \in \mathbb{C}$ with $\|A - zI\| \leq \|A - (z+c)I\|$ and $\|A - (z+c)I\| \geq 1$,*

$$r_\epsilon(A) + |c| \leq r_{\epsilon+|c|}(A).$$

Proof: Let $z \in \sigma_\epsilon(A)$ and $|z| = r_\epsilon(A)$. There exist some non-zero vector $u \in \mathbb{C}^N$, $E \in \mathbb{C}^{N\times N}$ with $\|E\| \leq \epsilon\|A - zI\|$ such that $(A+E)u = zu$ [1]. Then for any $c \in \mathbb{C}$

$$(A + E + cI)u = (z + c)u,$$

which means that $z + c \in \sigma(A + E + cI)$. Since

$$\|E + cI\| \leq \epsilon\|A - zI\| + |c| \leq \epsilon\|A - (z + c)I\| + |c|\|A - (z + c)I\|,$$

we have $z + c \in \sigma_{\epsilon+|c|}(A)$. Thus $r_\epsilon(A) + |c| \leq r_{\epsilon+|c|}(A)$. ■

The following theorem extends the result proved in Section 3 to almost commuting matrices.

Theorem 5.5: *Let $A, B \in BL(\mathbb{C}^N)$ and $0 < \epsilon < 1$. Further assume that $\|A\| \leq 1, \|B\| \leq 1$, $\|AB - BA\| \leq \delta$ for some $\delta \geq 0$. Then there exists a function $l(\delta, \epsilon)$ such that*

$$r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B) + 3\, l(\delta, \epsilon),$$
$$r_\epsilon(AB) \leq [r_\epsilon(A) + l(\delta, \epsilon)][r_\epsilon(B) + l(\delta, \epsilon)] + l(\delta, \epsilon)$$

and for each fixed ϵ, $l(\delta, \epsilon) \to 0$ whenever $\delta \to 0$.

Proof: From [12], there exists $f(\delta)$ such that $\|A - A'\| \leq f(\delta), \|B - B'\| \leq f(\delta)$ and $f(\delta) \to 0$ as $\delta \to 0$. To simplify the notation, we have suppressed the dependence of N on all functions [7]. Since the map $A \mapsto r_\epsilon(A)$ is continuous, equations (3)–(6) imply

$$|r_\epsilon(A) - r_\epsilon(A')| \leq l(\delta, \epsilon)$$
$$|r_\epsilon(B) - r_\epsilon(B')| \leq l(\delta, \epsilon)$$
$$|r_\epsilon(A + B) - r_\epsilon(A' + B')| \leq l(\delta, \epsilon)$$
$$|r_\epsilon(AB) - r_\epsilon(A'B')| \leq l(\delta, \epsilon)$$

for some $l(\delta, \epsilon)$ and for each fixed $0 < \epsilon < 1$ we have $l(\delta, \epsilon)$ going to zero whenever δ goes to zero. The last two assertions follows from the fact that the matrix addition and matrix multiplication are continuous operations. From (7) to (10), we have

$$r_\epsilon(A + B) \leq r_\epsilon(A) + r_\epsilon(B) + 3\, l(\delta, \epsilon).$$

From (11) to (14), we have

$$r_\epsilon(AB) \leq [r_\epsilon(A) + l(\delta, \epsilon)][r_\epsilon(B) + l(\delta, \epsilon)] + l(\delta, \epsilon).$$

We also have for each $0 < \epsilon < 1$, $l(\delta, \epsilon) \to 0$ whenever $\delta \to 0$. ■

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