A RELATIVELY ELEMENTARY PROOF OF A CHARACTERIZATION OF POLYNOMIALS

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1. Introduction

In his Presidential Address(Technical) to the Fifty Eighth Annual Conference of the Indian Mathematical Society in 1993(See [2] for the full text.), Prof. V. Kannan asked whether it is possible to give an elementary proof of the following characterization of polynomials among entire functions:
If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w)$ is a finite set for every $w \in \mathbb{C}$, then f must be a polynomial.

It is easy to prove this(actually a stronger version of this, see Remark 2.4) by making use of the Big (Great) Picard Theorem (See Corollary 12.4.4, p. 303 of Conway [1].). This Big Picard Theorem is usually not taught in the first introductory courses in complex analysis. The problem was to find a proof that does not use the Big Picard Theorem.

The aim of this note is to give such a " relatively elementary" proof. In fact, we prove a somewhat stronger version(See Corollary 2.3). We use the following tools: the Casorati-Weierstrass Theorem (Theorem 5.1.21 in [1]), the open mapping theorem for analytic functions (Theorem 4.7.5 in [1]) and the Baire category theorem. All these theorems are also nontrivial, but all these are usually taught in the first introductory courses in real analysis and complex analysis whereas the Big Picard Theorem is usually taught in an advanced course in complex analysis.

2. Elementary proof

We use the following notation: For $a \in \mathbb{C}$ and $r > 0$, the punctured disc with center at a and radius r is denoted by $D'(a, r)$. Thus,

$$D'(a, r) := \{z \in \mathbb{C} : 0 < |z - a| < r\}$$

Theorem 2.1. *Suppose Ω is an open set in $\mathbb{C}$, $a \in \Omega$ and f is an analytic function in $\Omega \setminus \{a\}$ with an essential singularity at a. Then there exists a dense subset D of $\mathbb{C}$, such that for every $w \in D$ and for every neighbourhood V of a, $f^{-1}(w) \cap V$ is an infinite set.*

Proof. Let V be a neighbourhood of a. There exists $n_0 \in \mathbb{N}$ such that $D'(a, \frac{1}{n}) \subseteq V$ for $n \geq n_0$. For such n, we define $D_n := f((D'(a, \frac{1}{n}))$. By the Casorati-Weierstrass Theorem, each D_n is dense in $\mathbb{C}$ and by the open mapping theorem each D_n is open. Let $D := \cap D_n$. Then D is dense in $\mathbb{C}$ by the Baire category theorem. Next let $w \in D$. Then $w \in D_n$ for all $n \geq n_0$, that is, there exists $z_n \in D'(a, \frac{1}{n})$ such that $f(z_n) = w$. It is easy to see that the set $\{z_n : n \geq n_0\}$ is infinite, because given any such z_n, one can find $m > n$ such that $\frac{1}{m} < |z_n - a|$. But then $|z_m - a| < \frac{1}{m} < |z_n - a|$. □

Remark 2.2. The Big Picard Theorem says that the set D in fact contains all complex numbers with one possible exception. In that sense, Theorem 2.1 may be called "Not So Big" or "Moderately Big" Picard Theorem.

Corollary 2.3. *If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w)$ is a finite set for every w in some nonempty open subset U of $\mathbb{C}$, then f must be a polynomial.*

Proof. Consider g defined by $g(z) = f(\frac{1}{z})$ for $z \in \mathbb{C} \setminus \{0\}$. If f is not a polynomial, then g has an essential singularity at 0. Let D be the dense subset of $\mathbb{C}$ given by Theorem 2.1. Then $D \cap U$ is nonempty. Let $w \in D \cap U$. Then $g^{-1}(w)$ is an infinite set. But then, $f^{-1}(w) = \{\frac{1}{z} : z \in g^{-1}(w)\}$ is an infinite set. This is a contradiction. □

Remark 2.4. It may be noted that by making use of the Big Picard Theorem, it is possible to prove a much stronger result:
If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f^{-1}(w_1)$ and $f^{-1}(w_2)$ are finite sets for two distinct complex numbers w_1 and w_2, then f must be a polynomial.
This is in fact Corollary 12.4.4 of [1]. It is not known whether this stronger result can be proved by elementary methods.

References

[1] J. B. Conway, *Functions of One Complex Variable*, Narosa Publishing House, New Delhi, 1973.
[2] V. Kannan, *Formulas for functions*, The Mathematics Student **62**(1993) 241 - 270.

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