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Three Types of Operator Monotonicity

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Abstract

A square real matrix A is called monotone if $Ax \geq 0 \Rightarrow x \geq 0$. Here $x = (x_i) \geq 0$ means that $x_i \geq 0$ for all i. Collatz has shown that the above is equivalent to the existence and nonnegativity of A^{-1}. In this paper an extension of monotonicity of operators between Hilbert spaces is presented. In particular, square monotonicity, rectangular monotonicity and semimonotonicity are characterized.

1 Introduction

A square real matrix A is called monotone if $Ax \geq 0 \Rightarrow x \geq 0$. Here $x = (x_i) \geq 0$ means that $x_i \geq 0$ for all i. Collatz [8] has shown that a matrix is monotone iff it is invertible and the inverse is nonnegative. The concept of monotonicity has been generalized in several ways. Rectangular real matrices were studied by Mangasarian [11] who has shown that a rectangular matrix is monotone iff it has a nonegative left inverse. These were extended to include characterizations of nonnegative Moore-Penrose inverses by Berman and Plemmons [5], [6] and by Werner [14]. Characterizations of nonnegative generalized inverses that satisfy the equation $TAT = T$ for a given A over partially ordered vector spaces were obtained by Sivakumar [12]. The approach there was purely algebraic. Later, nonegative group inverses were studied for operators on Hilbert spaces [13], where extensions of the notion of range monotonicity (See [6] for the finite matrix definition of range monotonicity) were also studied. Berman and Plemmons [7]

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have shown how monotonicity plays an important role in such diverse problems as convergence of iterative methods for linear systems, the theory of Markov chains, in linear programming problems and in linear economic models.

In this paper we continue the study of monotonicity and consider three fundamental types of monotonicity in the setting of (possibly) infinite dimensional Hilbert spaces. We organize the paper as follows. In section 2, after a review of the preliminary concepts we extend the notion of square monotonicity of finite matrices to operators on Hilbert spaces and present a generalization of Collatz's result. The main result in this section is the following.

Theorem A Let H be a real Hilbert space with a strict self-dual cone P and $A \in BL(H)$ with closed range. Then A is monotone and onto, iff A^{-1} exists and A^{-1} is positive, iff A and A^* are monotone.
In section 3, we study rectangular monotonicity where we present the following result which is an extension of Mangasarian's result (Theorem 3.4).

Theorem B Let H_1 and H_2 be real Hilbert spaces with cones P_1 and P_2, respectively with P_1 generating and P_2 self-dual. Suppose that there exists an orthonormal basis $\{u^\alpha : \alpha \in J\}$, J an index set, of H_1 with $u^\alpha \in P_1$ for all α. Let $A \in BL(H_1, H_2)$ with $R(A)$ and $N(A^*) + P_2$ both closed. Then A *is monotone* iff A has a nonnegative left inverse.
Finally, in section 4, we generalize the notion of semimonotonicity and give a characterization (Theorem 4.2).

2 Square Monotonicity

We first briefly review some of the concepts that will be used in the rest of the paper.

Definition 2.1 *A nonempty set P in a real vector spae V is called a cone if $\lambda P \subseteq P$ for all $\lambda \geq 0$ and P is convex. A cone P is said to be strict if $P \cap -P = \{0\}$. A cone P is said to be generating if $V = P - P$.*

Definition 2.2 *Let V be a real inner product space and P be a cone. The dual cone of P, denoted by P^*, is defined by*

$$P^* = \{y \in V : \langle x, y \rangle \geq 0,\ \forall\ x \in P\}.$$

The cone P is called self-dual if $P = P^*$.

Example 2.3 (i) *$\mathbb{R}^n_+$ the nonnegative orthant of $\mathbb{R}^n$, the $n-$ dimensional real Euclidean space is a self-dual closed generating cone.*

(ii) *Let $P = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_3 \geq 0, 2x_1x_3 \geq x_2^2\}$. Then P is a self-dual cone. This cone was studied in connection with investigations of solutions of linear systems over non-polyhedral cones [2] and is also well-known in classification of duality states of linear programming problems.*

(iii) *Let ℓ^2 denote the real Hilbert space of all square summable real sequences and $P = \{x \in \ell^2 : x_i \geq 0, \ \forall i\}$. Then P is a self-dual closed generating cone.*

In the rest of the paper we will assume that all cones P are nondegenerate, i.e., $P \neq \{0\}$. Let $BL(H_1, H_2)$ denote the space of bounded linear maps from H_1 into H_2. If $H_1 = H_2$, then $BL(H_1, H_2)$ will be denoted by $BL(H)$.

Definition 2.4 *Let P_1 and P_2 be cones in real vector spaces V_1 and V_2, respectively. A map $A : V_1 \to V_2$ is said to be positive relative to P_1, P_2 if $A(P_1) \subseteq P_2$. We will henceforth refer to such operators as positive.*

We begin with the following definition of monotonicity of operators between vector spaces.

Definition 2.5 *Let V_1 and V_2 be a real vector spaces with cones P_1 and P_2, respectively and $A : V_1 \to V_2$ be linear. We say that A is monotone relative to P_1, P_2, if $x \in V$ and $Ax \in P_2 \Rightarrow x \in P_1$. We will refer to such operators as monotone.*

We first show that a verbatim analogue of Collatz's result does not hold in infinite dimensional spaces.

Example 2.6 *Let $H = \ell^2$ with the usual cone P and $A : \ell^2 \to \ell^2$ be the right shift operator, i.e., $A(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$. Then A is monotone but A is not onto. Hence A^{-1} does not exist.*

Remark 2.7 Let V be a real vector space, $A : V \to V$ and P be a strict cone. If A is monotone it follows that A is one-one (also see remark 2.8.) It also follows that if A^{-1} exists and A^{-1} is positive then $Ax \in P \Rightarrow x = A^{-1}(Ax) \in P$, that is A is monotone. Further, if A is monotone and A^{-1} exists then A^{-1} is positive.

Remark 2.8 If the cone is not strict, then we may have A to be monotone without A being one-one. This is shown as follows: Let $P = \{x \in \mathbb{R}^2 : x_2 = 0\}$ and $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $Ax \in P \Rightarrow x \in P$. Clearly A is not one-one.

We now proceed to give a generalization of Collatz's result. We need the following simple lemma. For Hilbert spaces H_1, H_2 and $A \in BL(H_1, H_2)$, the adjoint A^* is defined by the equation $\langle Ax, y\rangle = \langle x, A^*y\rangle$, for all $x \in H_1, y \in H_2$.

Lemma 2.9 *Let H_1 and H_2 be a real Hilbert spaces with self-dual cones P_1 and P_2, respectively. Let $A \in BL(H_1, H_2)$. Then A is positive iff A^* is positive.*

Proof. Clearly, it is sufficient to establish the necessity part. Let A, be positive, $y \in P_2$ and $x = A^*y$. Let $u \in P_1$ be arbitrary. Then $\langle x, u\rangle = \langle A^*y, u\rangle = \langle y, Au\rangle \geq 0$, as $Au \in AP_1 \subseteq AP_2$. So $x \in P_1^* = P_1$. Thus A^* is positive. □

Theorem 2.10 *Let H be a real Hilbert space with a strict self-dual cone P and $A \in BL(H)$ with closed range. Then the following are equivalent:*

(i) *A is monotone and onto.*

(ii) *A^{-1} exists and A^{-1} is positive.*

(iii) *A and A^* are monotone.*

Proof. (i)⇒ (ii) if A is monotone and onto, then A is one-one and onto and so A^{-1} exists. Thus A^{-1} is positive, by Remark 2.7.
(ii)⇒ (iii) Follows from Remark 2.7 and Lemma 2.8.
(iii)⇒ (i) If A^* is monotone, then A^* is one-one and hence $R(A)$ is dense in H. Since $R(A)$ is closed it follows that A is onto. The proof is now complete. □

Corollary 2.11 (Collatz [8]) *Let A be a square matrix. Then all the statements in Theorem 2.10 are also equivalent to:*

(iv) *A is monotone.*

Proof. If A is monotone, then A is one-one and hence onto. □

Example 2.12 *Consider the operator $A : \ell^2 \to \ell^2$ defined by $A(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$. Then $A = A^*$ and is monotone. However, A^{-1} does not exist. Note that $R(A)$ is not closed. Thus the condition that $R(A)$ is closed is indispensable in Theorem 2.10.*

The next example illustrates Theorem 2.10

Example 2.13 *Let $H_1 = H_2 = \ell^2$ and $A \in BL(\ell^2)$ be defined by*

$$A(x_1, x_2, \ldots) = (x_1, -x_1 + x_2, x_3, -x_3 + x_4, x_5, -x_5 + x_6, \ldots).$$

Then A is monotone and onto. A^* is given by

$$A^*(x_1, x_2, \ldots) = (x_1 - x_2, x_2, x_3 - x_4, x_4, x_5 - x_6, x_6, \ldots).$$

It follows that A^* is monotone. Define $T \in BL(\ell^2)$ by

$$T(x_1, x_2, \ldots) = (x_1, x_1 + x_2, x_3, x_3 + x_4, x_5, x_5 + x_6, \ldots).$$

Then it can be shown that $T = A^{-1}$. Clearly, T is positive, viz., $x \geq 0 \Longrightarrow Tx \geq 0$.

In the following result, we collect a few properties of monotonicity in the finite dimensional case.

Proposition 2.14 *Let A and B be real square matrices. Then*

(i) *A is monotone $\Leftrightarrow$ A^T is monotone.*

(ii) *A and B monotone $\Rightarrow$ AB and BA are monotone.*

(iii) *A and B monotone with $A \geq B \Rightarrow A^{-1} \leq B^{-1}$, where $A \leq B$ means that the entries of $A - B$ are nonnegative.*

Proof. (i) By Corollary 2.11, *A is monotone* $\Leftrightarrow$ *A^{-1} exists and $A^{-1} \geq \mathbf{0}$ $\Leftrightarrow$ $(A^T)^{-1}$ exists and $(A^T)^{-1} \geq \mathbf{0} \Leftrightarrow A^T$ is monotone.* (ii) Straight forward. (iii) Follows from the identity $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$. □

Remark 2.15 In infinite dimensional spaces, *(i)* of proposition 2.14 is not true. This is illustrated by the right sift operator A on ℓ^2. Then A^* is the left shift operator and clearly A^* is not monotone. It is easy to verify that *(ii)* is true in infinite dimensional Hilbert spaces. However, *(iii)* is not true in the infinite setting. Let A be the identity operator on ℓ^2 and $B : \ell^2 \rightarrow \ell^2$ be defined by $B(x_1, x_2, \ldots) = (x_1, x_2/2, x_3/3, \ldots)$. Then $x \geq \mathbf{0} \Rightarrow (A - B)x \geq \mathbf{0}$ with the usual order. But B^{-1} does not exist. Note that $R(B)$ is not closed in ℓ^2. However, if in addition, A and B are bounded linear operators with closed ranges and are both onto, then *(iii)* holds. The proof of Proposition 2.14 (iii) simply extends in this case.

3 Rectangular Monotonicity

We next turn to the concept of rectangular monotonicity. The notion of monotonicity for square matrices was extended to the case of rectangular real matrices by Mangasarian [11], who has shown that the monotonicity of A is equivalent to the existence of a nonnegative left inverse of A. To prove a generalization of Mangasarian's result we need the following two results.

Theorem 3.1 (Ben-Israel and Charnes [3]) *Let H_1 and H_2 be real Hilbert spaces with H_1 partially ordered by the self-dual cone P_1. Let $A \in BL(H_1, H_2)$ with closed range. Suppose $N(A) + P_1$ is closed. Then either*

(i) $Ax = b, x \in P_1$ *has a solution*
or(exclusive)

(ii) $A^*y \in P_1,\ \langle y, b\rangle < 0$ *has a solution.*

Remark 3.2 Theorem 3.1 gives an extension of the Farkas' lemma to infinite dimensional spaces. The form of the above theorem is slightly different from the actual result in [3]. We have chosen this as this form is convenient. Also, the original theorem of Ben-Israel and Charnes holds in a more general setting of topological vector spaces.

Theorem 3.3 (Lomonosov, Corollary 2.5 in [1]) *Let X_1, X_2 be partially ordered Banach spaces with closed cones. If the cone of X_1 is also generating, then every positive operator from X_1 into X_2 is continuous.*

We now present the aforesaid generalization.

Theorem 3.4 *Let H_1 and H_2 be real Hilbert spaces with cones P_1 and P_2, respectively with P_1 generating and P_2 self-dual. Suppose that there exists an orthonormal basis $\{u^\alpha : \alpha \in J\}$, J an index set, of H_1 with $u^\alpha \in P_1$ for all α. Let $A \in BL(H_1, H_2)$ with $R(A)$ and $N(A^*) + P_2$ both closed. Then*

$$A \text{ is monotone } \Leftrightarrow \exists \text{ a positive } Y \in BL(H_2, H_1) \text{ such that } YA = I.$$

Proof. We only prove the necessity part. Let A be monotone. Then

$$Ax \in P_2 \Rightarrow x \in P_1.$$

Now, $x \in P_1$ and $u^\alpha \in P_1$ for all α would mean that

$$x = \sum_\alpha \langle x, u^\alpha\rangle, with \langle x, u^\alpha\rangle \geq 0 \ \forall \alpha.$$

Thus, if A is monotone then for any α,

$$Ax \in P_2,\ \langle x, u^\alpha\rangle < 0$$

has no solution. By Theorem 3.1 $A^*z = u^\alpha$ has a solution $w \in P_2^* = P_2$ for every α. Let z^α be a solution for each α and let $Z : H_1 \to H_2$ be defined by $Z(u^\alpha) = z^\alpha$ for all α. Then $A^*Z(u^\alpha) = A^*(z^\alpha) = u^\alpha$. Thus $A^*Z = I$. Clearly Z is positive. By Theorem 3.3, Z is bounded. Let $Y = Z^*$. Then by Lemma 2.9, Y is positive and $YA = I$. $\square$

Remark 3.5 It is important to observe that the standard cone in $\mathbb{R}^n$, namely $\mathbb{R}^n_+$, satisfies the property that $N(B) + \mathbb{R}^n_+$ is closed, for any $m \times n$ matrix B. This follows from the fact that $N(B) + \mathbb{R}^n_+$ is a polyhedral cone (Lemma 3.4, [2]). Thus, we have Mangasarian's result as a corollary to Theorem 3.4.

Corollary 3.6 (Mangasarian [11]) *If A is an $m \times n$ matrix which is monotone, then A has a nonnegative left inverse.*

Proof. Let $H_1 = \mathbb{R}^n, H_2 = \mathbb{R}^m$, $P_1 = \mathbb{R}^n_+$ and $H_2 = \mathbb{R}^m_+$. Then P_1 is generating, P_2 is self-dual, the standard basis of $\mathbb{R}^n$ satisfies the condition on P_1, while $N(A^*) + P_2$ is closed in view of the earlier remark. The other conditions of Theorem 3.4 always hold in finite dimensional spaces. $\square$

Remark 3.7 We show by an example that the condition that $N(A^*) + P_2$ be closed is indispensable, even in the finite dimensional case. Let $H_1 = \mathbb{R}^2, H_2 = \mathbb{R}^3$ with $P_1 = \mathbb{R}^2_+$ and $P_2 = \{x \in \mathbb{R}^3 : x_1 \geq 0, x_3 \geq 0, 2x_1x_3 \geq x_2^2\}$. Then P_1 is generating and P_2 is self-dual. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $Ax \in P_2 \Rightarrow x \in P_1$, so that A is monotone. It can be shown that the most general left-inverse of A is given by $Y = \begin{pmatrix} a & 0 & 1 \\ b & 1 & 0 \end{pmatrix}$, $a, b \in \mathbb{R}$. If Y is nonegative, then by taking $y^1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \in P_2$, (as Yy^1 must belong to P_1,) we must have $b \geq 1$. However, if we let $y^0 = \begin{pmatrix} 1/b \\ -2 \\ 2b \end{pmatrix}$, then $y^0 \in P_2$ with $Yy^0 = \begin{pmatrix} c \\ -1 \end{pmatrix} \notin P_1$, where c is some constant. Thus Y cannot be nonnegative. We next show that $N(A^*) + P_2$ is not closed. If we set $u^k = \begin{pmatrix} -k \\ 0 \\ 0 \end{pmatrix}$ and $v^k = \begin{pmatrix} k \\ 1 \\ 1/k \end{pmatrix}$, then for all k, $A^*u^k = 0$ and $v^k \in P_2$. Setting $z^k = u^k + v^k = \begin{pmatrix} 0 \\ 1 \\ 1/k \end{pmatrix}$, we see that z^k converges to $z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Next, if $z \in N(A^*) + P_2$, then $z = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}$, where $\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \in P_2, r \in \mathbb{R}$. As $t_3 = 0$, we have $t_2 = 0$, an absurdity. Thus $N(A^*) + P_2$ is not closed.

We next consider the question of when $N(A^*) + P_1$ will be closed. For the first characterization, we will need the notion of generalized inverses, which we review first. Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a linear map. Consider the following equations due to Penrose:

$$AXA = A$$

$$XAX = X$$
$$(AX)^* = AX$$
$$(XA)^* = XA.$$

It is known that [10], if A is bounded and $R(A)$ is closed, then there exists a unique bounded linear map $X : H_2 \to H_1$ that satisfies all the four equations. Such an X is called the Moore-Penrose inverse (or the pseduo inverse) and is denoted by $A^\dagger$. The following properties of $A^\dagger$ are also well known:

$$AA^\dagger = P_{R(A)};\ \ A^\dagger A = P_{R(A^*)},$$

where P_N denotes the orthogonal projection on N.

We now give a necessary and sufficient condition for the closedness of $N(B) + P_1$ for a bounded linear operator B and a closed cone P_1. This generalizes a corresponding result for finite dimensional spaces, due to Abrams. ([4], Lemma 3.1)

Theorem 3.8 *Let H_1 and H_2 be real Hilbert spaces with H_1 partially ordered by the positive cone P_1. Let $B \in BL(H_1, H_2)$ with $R(B)$ closed. Then $N(B) + P_1$ is closed in H_1 iff BP_1 is closed in H_2.*

Proof. Let BP_1 be closed. Let $z^k \in N(B) + P_1$. Then $z^k = y^k + x^k$, where $By^k = 0\ \forall k$, $x^k \in P_1\ \forall k$ and $Bx^k = Bz^k$. Suppose that $z^k \to z$. Then $Bz^k \to Bz$. We must show that $z \in N(B)+P_1$. Now, since BP_1 is closed it follows that there is $x \in P_1$ such that $Bz = Bx$, i.e., $z-x \in N(B)$. So $z = x+y, x \in P_1, y \in N(B)$. Thus $N(B) + P_1$ is closed.

Conversely, suppose that $N(B)+P_1$ is closed and that $x^k \in P_1$ such that $Bx^k \to b$. We must show that $b \in BP_1$. Let $x^k = y^k + z^k, y^k \in R(B^*), z^k \in N(B)$. Then $Bx^k = By^k$ and $y^k = B^\dagger By^k = B^\dagger Bx^k \to B^\dagger b$. Thus $y^k = x^k - z^k \to B^\dagger b$, so that $B^\dagger b \in N(B) + P_1$. Let $B^\dagger b = x - z$, where $x \in P_1, z \in N(B)$. Then $Bx = BB^\dagger b + Bz = BB^\dagger b = BB^\dagger(limBx^k) = lim(BB^\dagger Bx^k) = limBx^k = b$. □

We next present a sufficient condition under which BP_1 is closed. This specializes a result of Fisher and Jerome [9] to Hilbert spaces.

Theorem 3.9 *(Theorem 2.6, [9]) Let H_1 and H_2 be real Hilbert spaces and $B \in BL(H_1, H_2)$ with $R(B)$ closed and $N(B)$ finite dimensional. Let Y_α be a family of Hilbert spaces, $C_\alpha \in BL(H_1, Y_\alpha)$ and K_α be a family of closed convex subsets of Y_α, $\alpha \in J$, J an index set. If $U = \{x \in H_1 : C_\alpha(x) \in K_\alpha\ \forall\alpha\}$, then BU is closed if U is contained in an algebraic complement of $N(B)$.*

As an application of the above theorem, we have the following.

Example 3.10 *(Example 2.3, [9]) Let H_1 and H_2 be real Hilbert spaces and $B \in BL(H_1, H_2)$. Let C_α be a family of real continuous linear functionals on H_1 and let $r = (r_\alpha)$ be given, where J is an index set and $\alpha \in J$. Define $U = \{x \in H_1 : C_\alpha(x) \geq r_\alpha \ \forall \alpha\}$. Suppose that $\{x \in H_1 : C_\alpha(x) \geq 0 \ \forall \alpha\} \cap N(B) = \{0\}$. If U is nonempty, then it follows from the above theorem that BU is closed and convex in H_2.*

Example 3.11 *Let $H_1 = H_2 = \ell^2, B \in BL(\ell^2)$ be the left shift operator. Then $R(B)$ is closed. It follows easily that $BP = \{x \in H_1 : x_i \geq 0, i = 2, 3, ...\}$ is closed. Thus by Theorem 3.8, $N(A^*) + P$ is closed, where $A^* = B$ and hence A is the right shift operator on ℓ^2. A is monotone and all the other conditions of Theorem 3.4 are satisfied. Clearly, $BA = I$ and B is positive.*

Remark 3.12 Let A be as above and for $\alpha \in \mathbb{R}, \alpha \geq 0$, define $B_\alpha : \ell^2 \to \ell^2$ by $B_\alpha(x_1, x_2, \ldots) = (\alpha x_1 + x_2, x_3, \ldots)$. Then $B_\alpha A = I$ and that each B_α is positive. Thus positive left-inverses are not unique in general.

4 Semimonotonicity

In this last section, we consider the notion of semimonotonicity. We first give a definition of semimonotonicity of operators over infinite dimensional spaces.

Definition 4.1 *Let H_1 and H_2 be real Hilbert spaces with cones P_1 and P_2, respectively and $A \in BL(H_1, H_2)$. We say that A is semimonotone if $Ax \in P_2 + N(A^*)$ and $x \in R(A^*) \Rightarrow x \in P_1$.*

The following theorem characterizes semimonotonicity of operators.

Theorem 4.2 *Let $A \in BL(H_1, H_2)$ have closed range. Then the following are equivalent:*

(i) *$A^\dagger$ is positive.*

(ii) *A is semimonotone.*

(iii) *$Ax \in AA^\dagger P_2$ and $x \in R(A^*) \Rightarrow x \in P_1$.*

Proof. (i)$\Rightarrow$ (ii) Let $A^\dagger$ be positive and $Ax = u+v$ with $u \in P_2$ and $v \in N(A^*)$. If $x \in R(A^*) = R(A^\dagger)$, then $x = A^\dagger Ax = A^\dagger u \in A^\dagger P_2 \subseteq P_1$.

(ii)$\Rightarrow$ (i) Let $w \in P_2$ and (iii) hold. If $u = AA^\dagger w$ and $v = (I - AA^\dagger)w$, then $w = u + v$ and $v \in N(A^\dagger) = N(A^*)$. Now, $A(A^\dagger w) = u = w - v \in P_2 + N(A^*)$. Also, $A^\dagger w \in R(A^\dagger) = R(A^*)$. Thus $A^\dagger w \in P_1$. Hence $A^\dagger$ is positive.

(ii) $\Leftrightarrow$ (iii) Clearly, it is sufficient to prove:

$$Ax \in P_2 + N(A^*) \Leftrightarrow Ax \in AA^\dagger P_2.$$

Consider $Ax = u + v$, $u \in P_2$ and $v \in N(A^*)$. Then $Ax = AA^\dagger Ax = AA^\dagger u \in AA^\dagger P_2$. Conversely, if $Ax = AA^\dagger u$ for some $u \in P_2$, then $u = AA^\dagger Ax + y = Ax + y$, $y \in N(A^*)$. So $AA^\dagger u = Ax = u - y \in P_2 + N(A^*)$. □

Corollary 4.3 (Berman and Plemmons [5]) *For a real matrix A of order $m \times n$, the following are equivalent:*

(i) $A^\dagger$ *is nonnegative.*

(ii) A *is semimonotone.*

(iii) $Ax \in AA^\dagger \mathbb{R}^m_+$ *and* $x \in R(A^T) \Rightarrow x \geq \boldsymbol{0}$.

Remark 4.4 Berman and Plemmons [5] have used a generalization of the Farkas' lemma in proving their result. In our extension to infinite dimensional spaces, we have given a proof that does not use any result of Farkas' type. Finally, it should be observed that a verbatim analogue of Farkas' lemma to infinite dimensional spaces does not hold.

We next discuss some properties of semimonotonicity.

Remark 4.5 (i) of Proposition 2.14 holds for a bounded linear operator with closed range over a real Hilbert space. This follows from the fact that $(A^\dagger)^* = (A^*)^\dagger$.

The following example shows that Proposition 2.14 (ii) does not hold if monotonicity is replaced by semimonotonicity, even in the finite dimensional case.

Example 4.6 *Let* $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ *and* $B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. *Then A is monotone and B is semimonotone. It can be shown that* $(BA)^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ *and hence BA is not semimonotone.*

The following example shows that Proposition 2.14(iii) does not hold if monotonicity is replaced by semimonotonicity, even in the finite dimensional case.

Example 4.7 *Let* $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \geq \boldsymbol{0}$. *Then* $A^\dagger = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix} \geq \boldsymbol{0}$.

However, we have the following result.

Proposition 4.8 *Let H_1 and H_2 be real Hilbert spaces with self-dual cones P_1 and P_2, respectively, $A \in BL(H_1, H_2)$ be onto and $B \in BL(H_2, H_1)$ be one-one. If A and B are semimonotone and $A \geq B$ then $A^\dagger \leq B^\dagger$.*

Proof. If A is onto, then $A^\dagger = A^*(AA^*)^{-1}$ so that $AA^\dagger = I$ and if B is one-one, then $B^\dagger = (B^*B)^{-1}B^*$ so that $B^\dagger B = I$. Since A and B are semimonotone, $A^\dagger \geq \mathbf{0}$ and $B^\dagger \geq \mathbf{0}$. The proof now follows from the identity $B^\dagger - A^\dagger = B^\dagger(A - B)A^\dagger$. □

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