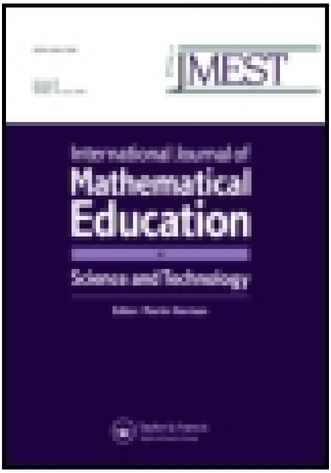

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Arzela-Ascoli theorem is stable

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Classroom notes

Arzela–Ascoli theorem is stable

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A quantitative version of the Arzela–Ascoli theorem is proved. This version implies that a closed and bounded subset of $C(X)$ is nearly compact, if and only if, it is nearly equicontinuous.

1. Introduction

We know that a process (a problem, an equation) is called stable if a small change (perturbation) in its input leads to a small change in the output. This idea is fairly common in the study of differential equations and many other operator equations. Applying this concept of stability to a mathematical theorem, it is natural to regard the hypotheses of the theorem as input and its conclusion as output. The crucial aspect in such an application is to decide what is meant by a small change. Jarosz [1] has given several illustrations of applying this idea to various theorems about Banach algebras.

In this note we give an illustration of the application of this concept to a well-known theorem in the undergraduate analysis, namely, the Arzela–Ascoli theorem. The importance of the Arzela–Ascoli theorem lies in the fact that it gives a characterization of compact subsets of $C(X)$. The importance of compactness in analysis is best explained by the famous quote from the classic article [2] of Hewit.

> A great many propositions in analysis are trivial for finite sets, true and reasonably simple for infinite compact sets; and either false or extremely difficult to prove for noncompact sets.

Thus it is of tremendous importance to know what are the compact sets in a concrete Banach space. Undoubtedly, the most popular characterization of compact sets is given by the Heine–Borel theorem which says that a subset of the real line is compact, if and only if, it is closed and bounded. In fact, this is true in every finite dimensional Banach space and is actually equivalent to finite dimensionality. Thus it is natural to ask: 'What additional properties should a closed and bounded set in an infinite dimensional Banach space have in order to be compact?' The Arzela–Ascoli theorem provides an answer to this question in the space $C(X)$ and the answer is: 'equicontinuity'. In other words, a closed and boundeed subset K of $C(X)$ is compact, if and only if, K is equicontinuous.

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We want to show that this theorem is stable, that is, a small change in the hypothesis of compactness produces only a small change in the property of equicontinuity and vice versa. That brings us to the crucial question: 'What is meant by a small change in compactness and a small change in equicontinuity?'

As far as compactness is concerned, this is already known. We recall this definition of 'a measure of non-compactness', define a similar 'measure of non-equicontinuity' and establish a relationship between the two.

2. Preliminaries

Let (X, d) be a metric space and M a subset of X. Define

$$\gamma(M) := \inf\{\epsilon > 0\colon M \text{ can be covered by a finite number of open balls with radius } \epsilon\}$$

$\gamma(M)$ is called the (Hausdorff) *measure of non-compactness* [3, 4]. If M is bounded, there exists $k > 0$ such that $d(x, y) \leqslant k$ for all $x, y \in M$. Obviously, $\gamma(M) \leqslant k$. $\gamma(M)$ can be strictly less than k. If $\gamma(M) = 0$, then for every $\epsilon > 0$, M can be covered by a finite number of open balls with radius ϵ. Such a set is called *totally bounded*. [5] (Some authors prefer to call this relatively compact.) A subset of a metric space is compact, if and only if, it is complete and totally bounded [5]. This is the motivation for the name *measure of non-compactness*. Perhaps more appropriate (and more clumsy!) name would have been *measure of non-totally-boundedness*! Note that in a finite dimensional Banach space every bounded set is totally bounded, hence $\gamma(M)$ is either 0 or ∞. On the other hand, if M is the unit ball in an infinite dimensional space, then $0 < \gamma(M) \leqslant 2$.

Next let X be a compact metric space and $C(X)$ be the Banach space of all complex-valued continuous functions defined on X equipped with the supremum norm, defined by,

$$\|f\| := \sup\{|f(x)|\colon x \in X\} \quad \text{for } f \in C(X)$$

Let A be a subset of $C(X)$. Define

$$\alpha(A) := \inf\{\epsilon > 0 : \text{there exists } \delta > 0 \quad \text{such that } d(x, y) < \delta \quad \text{implies} \quad |f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in X \quad \text{and for all } f \in A\}$$

We shall call $\alpha(A)$, the *measure of non-equicontinuity*. As in the case of γ, it is easy to see that $\alpha(A)$ is finite, whenever A is bounded. Further, $\alpha(A) = 0$, if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$ and for all $f \in A$. As is well known, such a family A of functions is called an *equicontinuous* family of functions and this is the motivation for the term *measure of non-equicontinuity*.

3. Main theorem

Theorem. Let X be a compact metric space and $A \subset C(X)$ be bounded. Then,

(i) $\alpha(A) \leqslant 2\gamma(A)$,
(ii) $\gamma(A) \leqslant 2\alpha(A)$.

Equivalently, $\frac{1}{2}\alpha(A) \leqslant \gamma(A) \leqslant 2\alpha(A)$.

Proof. Let $\alpha := \alpha(A)$ and $\gamma := \gamma(A)$, where $\alpha(A)$ is the measure of non-equicontinuity of A and $\gamma(A)$ is the measure of non-compactness of A as defined above.

Let $\epsilon > 0$. A can be covered by a finite number of open balls with radius $\gamma + \epsilon$ and centres at, say, $f_1, \ldots, f_n$. Since each f_k is uniformly continuous, there exists $\delta_k > 0$ such that $d(x, y) < \delta_k$ implies $|f_k(x) - f_k(y)| < \epsilon$. Let $\delta := \min\{\delta_k, k = 1, \ldots, n\}$. Now let $f \in A$. There exists f_k such that $\|f - f_k\| < \gamma + \epsilon$. Then $d(x, y) < \delta$ implies

$$\begin{aligned}|f(x) - f(y)| &\leqslant |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &< \gamma + \epsilon + \epsilon + \gamma + \epsilon = 2\gamma + 3\epsilon\end{aligned}$$

Thus $\alpha \leqslant 2\gamma + 3\epsilon$. Since ϵ was arbitrary, $\alpha \leqslant 2\gamma$. This proves (i).

To prove (ii), again let $\epsilon > 0$. There exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \alpha + \epsilon$ for all $x, y \in X$ and for all $f \in A$. Let $V_x := \{y \in X : d(x, y) < \delta\}$. Then $y \in V_x$ implies $|f(x) - f(y)| < \alpha + \epsilon$ for all $f \in A$. X, being compact, is covered by a finite number of such sets, say, $V_{x_1}, \ldots, V_{x_m}$. Consider the set $N := \{(f(x_1), \ldots, f(x_m)) : f \in A\} \subseteq \mathbf{C^m}$. Since A is bounded in $C(X)$, N is bounded in $\mathbf{C^m}$ and is hence totally bounded in $\|.\|_\infty$ norm. Hence N is covered by a finite number of open balls with radius ϵ and centres at, say, $\{(f_1(x_1), \ldots, f_1(x_m)), \ldots, (f_k(x_1), \ldots, f_k(x_m))\}$. Now let $f \in A$. Then there exists f_i such that $|f(x_j) - f_i(x_j)| < \epsilon$ for $j = 1, \ldots, m$. Let $x \in X$. Then $x \in V_{x_p}$ for some p. Now

$$\begin{aligned}|f(x) - f_i(x)| &\leqslant |f(x) - f(x_p)| + |f(x_p) - f_i(x_p)| + |f_i(x_p) - f_i(x)| \\ &< \alpha + \epsilon + \epsilon + \alpha + \epsilon = 2\alpha + 3\epsilon\end{aligned}$$

that is, $\|f - f_i\| < 2\alpha + 3\epsilon$. Thus A is covered by k open balls with radius $2\alpha + 3\epsilon$. Hence $\gamma \leqslant 2\alpha + 3\epsilon$. Since ϵ was arbitrary, we have $\gamma \leqslant 2\alpha$. □

Corollary (*Arzela–Ascoli theorem*). Let A be a closed and bounded subset of $C(X)$. Then A is compact, if and only if, A is equicontinuous.

Proof. Since A is closed, it is complete. Hence A is compact, if and only if, it is totally bounded, if and only if, $\gamma(A) = 0$, if and only if, $\alpha(A) = 0$, if and only if, A is equicontinuous. □

Remark. Interested readers may investigate whether characterizations of compactness in other infinite dimensional Banach spaces as given in Table IV pp. 374–379 of [6] are *stable* in our sense. See also [7].

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Convergence of Newton's and Halley's methods in the complex plane

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In this note the convergence behaviour of both Newton's and Halley's methods for a general quadratic function in the complex plane is investigated. It is concluded that both methods exhibit similar convergence behaviour in this case.

1. Introduction

Newton's method and Halley's method are two of the most classical iterative methods for finding the approximation for a zero of a real or complex function ([1–5] and references therein). Both Newton's method and Halley's method converge if the initial guess is close enough to the target zero of the function. If the target zero is a simple zero of the function, then the convergence rate for Newton's method is quadratic, while it is cubic for Halley's method. But, when the initial guess is not close enough to the target zero, both Newton's and Halley's methods often diverge and even go into chaotic behaviour, especially in the complex plane case. In this note, we investigate the convergence behaviour of both Newton's and Halley's methods for a general quadratic function in the complex plane such as

$$f(z) = c(z - c_1)(z - c_2) \tag{1}$$

where c is the complex leading coefficient of $f(z)$ with two complex zeros c_1 and c_2. We show that both Newton's and Halley's methods have similar convergence behaviour for $f(z)$ in equation (1). And the corresponding behaviours exhibit quadratic convergence rate for Newton's method and cubic convergence rate for Halley's method if $c_1 \neq c_2$. When $f(z)$ has a double zero, that is, when $c_1 = c_2$, then the corresponding convergence behaviours indicate that both Newton's and Halley's methods converge linearly.

2. Convergence of Newton's method

The iteration formula for Newton's method is given by