SPECTRAL ISOMETRIES OF REAL COMMUTATIVE BANACH ALGEBRAS

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Abstract. For a Banach algebra A, $rad(A)$ denotes the radical of A and for $a \in A$, $r(a)$ denotes the spectral radius of A. The following theorem is proved: Let A and B be real commutative Banach algebras, each with the unit 1. Let $T : A \to B$ be a linear map satisfying $T(1) = 1$, T is onto and $r(T(a)) = r(a)$ for all $a \in A$. Then T is a homomorphism modulo $rad(B)$. (This means that $T(ab) - T(a)T(b) \in rad(B)$ for all $a, b \in A$). If in addition, B is semisimple, then T is a homomorphism. Further, if A is also semisimple, then T is an isomorphism.

As a consequence, we get the following: Let X, Y be compact Hausdorff spaces, let $C(X)$, $C(Y)$ denote the Banach algebras of all complex valued continuous functions on X, Y respectively each with the supremum norm. Let A, B be uniformly closed real subalgebras of $C(X)$, $C(Y)$ respectively, each containing the constant function 1. Let $T : A \to B$ be a linear onto map. Then T is an isometry and $T(1)$ = 1 if and only if T is an algebra isomorphism.

These results generalize the classical theorems of Banach-Stone and Nagasawa.

Introduction

For a compact Hausdorff space X, $C(X)$ denotes the Banach algebra of all complex valued continuous functions defined on X with the point wise operations and supremum norm, defined by

$$\|f\| := sup\{|f(x)|, x \in X\}, \quad f \in C(X)$$

The classical Banach-Stone theorem states that for compact Hausdorff spaces X and Y if $C(X)$ and $C(Y)$ are linearly isometric, then they are isomorphic as algebras.

Several generalizations of this well known theorem are available in the literature [1, 2, 5]. Some generalizations of this theorem are provided by replacing $C(X)$ by its subspaces or subalgebras. A subset A of $C(X)$ is said to separate points of X, if for all $x, y \in X$ with $x \neq y$, there exists $f \in A$, such that $f(x) \neq f(y)$. In 1959, Nagasawa [8] extended the Banach-Stone theorem for the complex function algebras, that is, for the uniformly closed subalgebras of $C(X)$ which contain the unit of $C(X)$ and separate the points of X. He proved that two function algebras are isomorphic as algebras if and only if they are linearly isometric as Banach spaces.

In 1991, Kulkarni and Arundhathi [6] have extended Nagasawa's theorem to real function algebras [7] which are certain uniformly closed real subalgebras of $C(X)$. (For definitions see Definition 1.2.)

In this paper we use the Gelfand theory of commutative real Banach algebras due to Ingelstam [4], [7] and prove that if certain commutative real Banach algebras are spectrally isometric (Definition 2.1) then they are isomorphic as algebras (Theorem 2.2). This result generalizes theorem of Nagasawa [8], and Kulkarni and Arundhathi (See Theorem 5.1.4 and Remark 5.1.5 in [7]).

1. Preliminaries and Notations

In this section we recall some basic concepts in the theory of real Banach algebras [4, 7]. An **algebra** A over a field $\mathbb{F}$ is a ring which is also a vector space over $\mathbb{F}$ such that $(\alpha a)(b) = \alpha(ab) = a(\alpha b)$ for all $a, b \in A$ and $\alpha \in \mathbb{F}$.
A is said to be **commutative** if $ab = ba$ for $a, b \in A$.

An algebra A is called **real algebra** if $\mathbb{F} = \mathbb{R}$, the field of real numbers, and a **complex algebra** if $\mathbb{F} = \mathbb{C}$, the field of complex numbers; such an algebra is said to be normed by $\|.\|$ if $(A, \|.\|)$ is a normed linear space such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.
A complete normed algebra is called a **Banach algebra**.
Note that every complex Banach algebra A can be considered as a real Banach algebra.
If A has a unit element, we denote it by 1.
Let A be a real algebra with unit 1 and $a \in A$. Then the **spectrum** of a, denoted by $Sp(a)$, is defined as follows:

$$Sp(a) := \{\lambda = s + it \in \mathbb{C} : (a - s)^2 + t^2 \text{ is not invertible in } A\}$$

(This definition of spectrum of an element of a real algebra is due to Kaplansky).
The **spectral radius** of an element $a \in A$ denoted by $r(a)$ is defined as $r(a) = \sup\{|\lambda| : \lambda \in Sp(a)\}$. If A is a Banach algebra, then $r(a)$ is finite and is given by the spectral radius formula: $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.
Let A and B be real or complex algebras. A linear map $T : A \to B$ is called an **algebra homomorphism** if $T(ab) = T(a)T(b)$ for all $a, b \in A$. Further if T is bijective then it is called an **algebra isomorphism**.
For a real commutative Banach algebra A with unit 1, the **Carrier space** of A denoted by $Car(A)$ is the set of all non-zero algebra homomorphisms of A into $\mathbb{C}$. For $a \in A$, define a mapping $\hat{a} : Car(A) \to \mathbb{C}$ by $\hat{a}(\phi) = \phi(a)$ for all $\phi \in Car(A)$, $\hat{a}$ is called the **Gelfand transform** of a. The weakest topology on $Car(A)$ which makes $\hat{a}$ continuous for all $a \in A$ is called the **Gelfand topology** on $Car(A)$.

Since A is a real algebra, $\bar{\phi} \in Car(A)$ whenever $\phi \in Car(A)$ and there is a natural map $\tau_A : Car(A) \to Car(A)$ defined by $\tau_A(\phi) = \bar{\phi}$. Also if $\phi \in Car(A)$ then $ker\ \phi = \{a \in A : \phi(a) = 0\}$ is a maximal ideal of A. In fact every maximal ideal of A is equal to $ker\ \phi$ for some $\phi \in Car(A)$. The **radical** of A denoted by $rad(A)$ is the intersection of all maximal ideals of A. It is well known that for a commutative Banach algebra A,
$rad(A) = \{a \in A : \hat{a} = 0\}$ [3].
A is said to be **semi-simple** if $rad(A) = \{0\}$.

The following theorem summarizes some of the properties of $Car(A)$, proof of which may be found in [4, 7]

Theorem 1.1 (See Theorem 1.2.9 of [7]). *Let A be a real commutative Banach algebra with unit 1.*

(1) *$Car(A)$ endowed with the Gelfand topology is a compact Hausdorff space and the mapping $\tau_A : Car(A) \to Car(A)$, defined by $\tau_A(\phi) = \bar{\phi}$, for all $\phi \in Car(A)$ is a homeomorphism such that $\tau_A^2 = \tau_A \circ \tau_A$ is the identity map on $Car(A)$.*
(2) *The mapping $a \to \hat{a}$, $a \in A$ is an algebra homomorphism of A into $C(Car(A))$.*
(3) *For $a \in A$, $\hat{a}(Car(A)) = Sp(a)$ and hence*

$$\|\hat{a}\| = \sup\{|\hat{a}(\phi)| : \phi \in Car(A)\} = r(a).$$

Let X be a compact Hausdorff space and $\tau : X \to X$ be a homeomorphism such that $\tau^2 = \tau o \tau$ is the identity map on X. Let

$$C(X, \tau) := \{f \in C(X) : f(\tau(x)) = \overline{f(x)} \text{ for all } x \in X\}.$$

Then $C(X, \tau)$ is a real Banach algebra. It is not a complex algebra. Note that if τ is the identity map on X, then $C(X, \tau)$ is an algebra of all continuous real valued functions on X.

Definition 1.2. A **real function algebra** on (X, τ) is a uniformly closed subalgebra of $C(X, \tau)$ which separates points of X and contains the constant function 1.

In [7], it is shown that every complex function algebra of $C(X)$ can be regarded as a real function algebra on (Y, τ) for some suitably chosen compact Hausdorff space Y and a topological involution τ on Y.

In [6], Kulkarni and Arundhathi have proved that two real function algebras are linearly isometric if and only if they are isomorphic as algebras. We refer to this result as real analogue of Nagasawa's theorem (See Theorem 5.1.4 in [7]).

Corollary 1.3. *Let A be a commutative real Banach algebra with unit 1 and $\hat{A} := \{\hat{a} : a \in A\}$. Then the closure of $\hat{A}$ (denoted by $cl(\hat{A})$) in $C(Car(A))$ is a real function algebra on $(Car(A), \tau_A)$.*

Proof. By Theorem 1.1, it follows that $\hat{A}$ is a real subalgebra of $C(Car(A))$. In fact, for $a \in A$ and $\phi \in Car(A)$, $(\hat{a} \circ \tau_A)(\phi) = \hat{a}(\bar{\phi}) = \bar{\phi}(a) = \bar{\hat{a}}(\phi)$. Therefore $\hat{a} \in C(Car(A), \tau_A)$ for all $a \in A$ and hence $\hat{A}$ is a real subalgebra of $C(Car(A), \tau_A)$.

Also, if $\phi_1, \phi_2 \in Car(A)$ and $\phi_1 \neq \phi_2$, then $\phi_1(a) \neq \phi_2(a)$ for some $a \in A$. Therefore $\hat{a}(\phi_1) \neq \hat{a}(\phi_2)$ and hence $\hat{A}$ separates points of $Car(A)$. Further, since each $\phi \in Car(A)$ is a non-zero algebra homomorphism, $\phi(1) = 1$ for all $\phi \in Car(A)$. Therefore, $\hat{1}$ is the constant function 1 on $Car(A)$. Hence $cl(\hat{A})$ in $C(Car(A))$ is a real function algebra on $(Car(A), \tau_A)$. □

2. The Main Theorem

Definition 2.1. Let A, B be Banach algebras each with unit 1. A linear map $T : A \to B$ is called a **spectral isometry** if $r(T(a)) = r(a)$ for all $a \in A$. It is called an **isometry** if $\|T(a)\| = \|a\|$ for all $a \in A$.

Theorem 2.2. *Let A, B be real commutative Banach algebras with unit 1 and $T : A \to B$ be an onto linear spectral isometry such that $T(1) = 1$. Then T is an algebra homomorphism modulo $rad(B)$. If in addition, B is semisimple, then T is a homomorphism. Further, if A is also semisimple, then T is an isomorphism. Conversely, if T is an isomorphism of A onto B, then T is a spectral isometry such that $T(1) = 1$.*

Proof. Recall from Theorem 1.1, that $\|\hat{a}\| = r(a)$, $\|\hat{b}\| = r(b)$ for all $a \in A$, $b \in B$. Define,

$$\hat{T} : \hat{A} \to \hat{B} \text{ by } \hat{T}(\hat{a}) = \widehat{T(a)}.$$

Claim: $\hat{T}$ is well defined.
Suppose $a_1, a_2 \in A$ such that $\hat{a_1} = \hat{a_2}$, i.e $(\widehat{a_1 - a_2}) = 0$ which implies $r(a_1 - a_2) = r(T(a_1 - a_2)) = \|T(\widehat{a_1 - a_2})\| = 0$ which implies that $\|\widehat{T(a_1)} - \widehat{T(a_2)}\| = 0$. Therefore $\hat{T}(\hat{a_1}) = \hat{T}(\hat{a_2})$.
This proves the claim. Clearly $\hat{T}$ is linear and for all $\hat{a} \in \hat{A}$

$$\|\hat{T}(\hat{a})\| = \|\widehat{T(a)}\| = r(T(a)) = r(a) = \|\hat{a}\|.$$

Therefore $\hat{T}$ is an isometry. Also $\hat{T}(\hat{1}) = \hat{1}$. Now, $\hat{T}$ is onto since T is onto. Therefore $\hat{T}$ is a unit preserving linear isometry of $\hat{A}$ onto $\hat{B}$.
Now, $\hat{T}$ can be extended linearly and isometrically to T' from $cl(\hat{A})$ onto $cl(\hat{B})$. Also, by Corollary 1.3, $cl(\hat{A})$ is a real function algebra on $(Car(A), \tau_A)$ and $cl(\hat{B})$ is a real function algebra on $(Car(B), \tau_B)$. Therefore T' is a unit preserving linear isometry of the real function algebra $cl(\hat{A})$ onto $cl(\hat{B})$.

Hence by the real analogue of Nagasawa's theorem, (Theorem 5.1.4 in [7]) T' is an algebra isomorphism. Therefore $T'/_{\hat{A}} = \hat{T}$ is also an algebra isomorphism, that is,

$$\hat{T}(\hat{a_1}.\hat{a_2}) = \hat{T}(\hat{a_1}).\hat{T}(\hat{a_2}).$$

Thus

$$(T(a_1.a_2) \widehat{- T(a_1)}.T(a_2)) = 0 \text{ for all } a_1, a_2 \in A.$$

i.e $T(a_1.a_2) - T(a_1).T(a_2) \in rad(B)$. Thus T is an algebra homomorphism modulo $rad(B)$. In addition if B is semisimple, that is, $rad(B) = \{0\}$ then clearly $T(a_1.a_2) = T(a_1).T(a_2)$ for all $a_1, a_2 \in A$. Therefore, T is an algebra homomorphism. Further, if A is also semisimple then $T(a) = 0$ implies $r(T(a)) = r(a) = \|\hat{a}\| = 0$ which implies $a \in rad(A)$ and hence $a = 0$. Therefore T is one-one. Hence T is an algebra isomorphism.

Conversely if T is an algebra isomorphism then $Sp(T(a)) = Sp(a)$ for all $a \in A$ (See Corollary 1.1.21 and Remark 1.1.22 of [7]) and hence T is a spectral isometry. □

Corollary 2.3. *Let X, Y be compact Hausdorff spaces, A and B be uniformly closed real subalgebras of $C(X)$, $C(Y)$ respectively, each containing the constant function 1. Let $T : A \to B$ be a linear onto map. Then T is an isometry and $T(1) = 1$ if and only if T is an algebra isomorphism.*

Proof. Since $\|f\| = r(f)$ for all $f \in A$ (respectively for all $f \in B$), A and B are semisimple and if $T : A \to B$ is a isometry, then T is an spectral isometry. Hence the corollary follows from Theorem 2.2. □

Remark 2.4. Note that we cannot replace spectral isometry by isometry in Theorem 2.2 as the following example shows. (See Example 1.4.6 in [7].) Let $A = (\mathbb{R}^2, || \, ||_\infty)$ with coordinate wise multiplication. For a fixed t with $0 \leq t \leq \frac{1}{2}$, let $A_t = (\mathbb{R}^2, || \, ||_\infty)$ with the multiplication $*_t$ defined by

$$(x_1, x_2) *_t (y_1, y_2) = (x_1 y_1 - t(x_1 - x_2)(y_1 - y_2), x_2 y_2 - t(x_1 - x_2)(y_1 - y_2)).$$

Then A_t is a commutative Banach algebra with unit $(1, 1)$. Let $T : A \to A_t$ be the identity map. Then T is a linear isometry of A onto A_t, A is semisimple, but T is not a homomorphism. Note that T is not a spectral isometry. For $a = (1, -1) \in A$, $r(a) = 1$. If $t = \frac{1}{4}$, $r(T(a)) = 0$.

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