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Spectrum and related sets: a survey

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Abstract

In order to understand the behaviour of a square matrix or a bounded linear operator on a Banach space or more generally an element of a Banach algebra, some subsets of the complex plane are associated with such an object. Most popular among these sets is the spectrum $\sigma(a)$ of an element a in a complex unital Banach algebra A with unit 1 defined as follows:

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \text{ is not invertible in } A\}.$$

Here and also in what follows, we identify $\lambda.1$ with λ. Also quite popular is Numerical range $V(a)$ of a. This is defined as follows:

$$V(a) := \{\phi(a) : \phi \text{ is a continuous linear functional on } A \text{ satisfying } \|\phi\| = 1 = \phi(1)\}.$$

Then there are many generalizations, modifications, approximations etc. of the spectrum. Let $\epsilon > 0$ and n a nonnegative integer. These include $\epsilon-$ condition spectrum $\sigma_\epsilon(a)$, $\epsilon-$pseudospectrum $\Lambda_\epsilon(a)$ and $(n,\epsilon)-$pseudospectrum $\Lambda_{n,\epsilon}(a)$. These are defined as follows:

$$\sigma_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|\lambda - a\|\|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\right\}$$

In this and the following definitions we follow the convention : $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

$$\Lambda_\epsilon(a) := \left\{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geq \frac{1}{\epsilon}\right\}$$
$$\Lambda_{n,\epsilon}(a) := \left\{\lambda \in \mathbb{C} : \|(\lambda - a)^{-2^n}\|^{1/2^n} \geq \frac{1}{\epsilon}\right\}.$$

In this survey article, we shall review some basic properties of these sets, relations among these sets and also discuss the effects of perturbations on these sets and the

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question of determining the properties of the element a from the knowledge of these sets.

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1 Introduction

Problems about linear systems of equations formulated as operator equations, approximate solutions of such equations, the eigen-value problems are some of the important problems of Linear Algebra. In many practical situations it becomes essential to pose these problems in an infinite dimensional setting. Functional Analysis plays a vital role in analysis of such problems. In recent years, it has been observed by some researchers such as Arveson, Bottcher and others that methods involving Banach algebra techniques can be quite useful in dealing with such problems. For example, see the classical books [4, 5]. In particular, methods of approximating an infinite dimensional problem by a sequence of finite dimensional problems, such as a finite section method, work very well in case of certain operators and fail dramatically in case of certain other operators. It was pointed out by Arveson [1] that a success or failure of such a method depends on whether the operator under consideration belongs to a particular Banach algebra. It also turns out that many problems in Analysis, Operator theory and Numerical Analysis are equivalent to the problem of determining whether a particular element in a suitably chosen Banach algebra is invertible or not (see [10]). For example, whether a given bounded linear operator on a Banach space is a Fredholm operator or not is equivalent to whether the corresponding element in a quotient algebra (known as Calkin algebra) is invertible or not. Similar equvivalence can be established between the problem of determining whether some approximation method works or not and the problem of determining the invertiblity of an associated element in a Banach algebra.

Closely related to the problem of invertiblity of an element of a Banach algebra is the concept of spectrum. The spectrum is a very useful concept in several applications. In concrete cases, it has well known interpretations such as spectrum of a square matrix or spectrum of an operator. Thus computation of the spectrum of an element is an important task. On the other hand, it is well known that the map $a \mapsto \sigma(a)$, that takes an element a of a Banach algebra A to its spectrum $\sigma(a)$, is not continuous in general. There are many examples in the literature to demonstrate this. In particular, if T is a bounded linear operator defined on a separable Hilbert space H with an orthonormal basis $\{e_j\}$, then the spectrum $\sigma(T)$ of T depends discontinuously on the matrix entries $\langle Te_j, e_i \rangle$. We have given one such example (see Example 8.1).

In order to overcome this difficulty of discontinuity, the researchers have suggested computation of some other sets than the spectrum, though the main objective may be the computation of the spectrum. The basic idea is that these sets should on the one hand provide approximation of the spectrum in some sense and at the same time should depend continuously on the elements under consideration at least in many cases of practical interest.

In the next six sections, we review the basic properties and some examples of these six sets, namely, Spectrum (Sect. 2), Ransford spectrum (Sect. 3), Numerical Range (Sect. 4), Condition spectrum (Sect. 5), ϵ-pseudospectrum (Sect. 6) and (n, ϵ)-pseudospectrum (Sect. 7). We also disccuss some relations among these sets. Section 8 deals with the question of stability of these sets. Finally in the last section we make a passing mention of some results about these sets without going into a detailed discussion. In general, proofs are not given, but references where these proofs can be found are cited.

We shall use the following notations throughout this article. Let

- $B(w; r) := \{z \in \mathbb{C} : |z - w| < r\}$, the open disc with the centre at w and radius r,
- $D(z_0; r) := \{z \in \mathbb{C} : |z - z_0| \leq r\}$, the closed disc with the centre at z_0 and radius r,
- $A + D(0; r) = \bigcup_{a \in A} D(a; r)$ for $A \subseteq \mathbb{C}$ and $d(z, K) = \inf\{|z - k| : k \in K\}$, the distance between a complex number z and a closed set $K \subseteq \mathbb{C}$.
- Let $\delta\Omega$ denote the boundary of a set $\Omega \subseteq \mathbb{C}$.
- $\mathbb{C}^{n \times n}$ denotes the space of square matrices of order n and $B(X)$ denotes the set of bounded linear operators on a Banach space X.

2 Spectrum

We shall review some basic concepts about spectrum in this section. Since our main objects of study are spectra of elements in a Banach algebra, we shall begin with some definitions related to a Banach algebra.

Definition 2.1 *Complex Algebra*: A *complex algebra* A is a ring that is also a complex vector space such that

$$(\alpha a)b = \alpha(ab) = a(\alpha b) \quad \text{for all} \quad a, b \in A,\ \alpha \in \mathbb{C}$$

A is called *commutative* if $ab = ba$ for all $a, b \in A$.

We shall assume that A has a unit element 1 satisfying $1a = a = a1$ for all $a \in A$. It is well known that when such a unit element exists, it is unique.

We need the following concepts.

Definition 2.2 **Invertibility** Let A be a complex algebra with the unit element 1. An element $a \in A$ is said to be *invertible* in A if there exists $b \in A$ such that $ab = 1 = ba$. Such an element b is called *inverse* of a. Also $a \in A$ is said to be *left invertible* in A if there exists $b \in A$ such that $ba = 1$. Such an element b is called a *left inverse* of a. Similarly, $a \in A$ is said to be *right invertible* in A if there exists $c \in A$ such that $ac = 1$. Such an element c is called a *right inverse* of a.

Remark 2.3 It is well known that if $a \in A$ is invertible, then it has a unique inverse and we shall denote it by a^{-1}. On the other hand, left or right inverse, even if exists, need not be ubique.

Definition 2.4 **Banach Algebra**: Let A be a complex algebra. An *algebra norm on* A is a function $\|.\| : A \to \mathbb{R}$ satisfying:

1. $\|a\| \geq 0$ for all $a \in A$ and $\|a\| = 0$ if and only if $a = 0$.
2. $\|\alpha a\| = |\alpha|\|a\|$ for all $a \in A$ and $\alpha \in \mathbb{R}$
3. $\|a + b\| \leq \|a\| + \|b\|$ for all $a, b \in A$.
4. $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$.

A *complex normed algebra* is a complex algebra A with an algebra norm defined on it. A *Banach algebra* is a complete normed algebra.

We shall assume that A is *unital*, that is A has unit 1 with $\|1\| = 1$.

Example 2.5 Let X be a compact Hausdorff space, and let $C(X)$ denote the set of all complex valued continuous functions on X. Then $C(X)$ is a commutative Banach algebra under pointwise operations and the sup norm given by

$$\|f\| := \sup\{|f(x)| : x \in X\}, \quad f \in C(X)$$

Example 2.6 Let H be a complex Hilbert space and let $BL(H)$ denote the set of all bounded(continuous) linear operators on H. Then $BL(H)$ is a Banach algebra under the usual operations and the operator norm given by

$$\|T\| := \sup\{\|T(x)\| : x \in H, \|x\| \leq 1\}, \quad T \in BL(H)$$

When H is of dimension n, $BL(H)$ can be identified with $\mathbb{C}^{n \times n}$, the algebra of all matrices of order $n \times n$ with complex entries.

More examples and basic theory of Banach algebras can be found in the following books [3, 25].

Definition 2.7 **Spectrum**: Let A be a complex Banach algebra with unit 1. For $\lambda \in \mathbb{C}$, $\lambda.1$ is identified with λ. Let $\mathrm{Inv}(A) = \{x \in A : x$ is invertible in $A\}$ and $\mathrm{Sing}(A) = \{x \in A : x$ is not invertible in $A\}$. The *spectrum* of an element $a \in A$ is defined as:

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \mathrm{Sing}(A)\}$$

The spectral radius of an element a is defined as:

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

The complement of the spectrum of an element a is called the *resolvent set of* a and is denoted by $\rho(a)$.

Thus when $A = C(X)$ and $f \in A$, $\sigma(f)$ coincides with the range of f.
Similarly when $A = \mathbb{C}^{n \times n}$ and $M \in A$, $\sigma(M)$ is the set of all eigenvalues of A.

We recall a few well known properties of the spectrum in the following theorem.

Theorem 2.8 *Let A be a complex Banach algebra with unit 1 and let $a \in A$. Then*

1. *$\sigma(a)$ is a nonempty compact subset of $\mathbb{C}$.*
2. The Spectral Radius Formula:

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

3. *The map $a \to \sigma(a)$ is upper semicontinuous.*

A proof of this theorem can be found in [3].

3 Ransford spectrum

The idea of spectrum has undergone many genrealizations. Ransford [28] gave a unified approach to many of these generalizations of spectrum. Though Ransford studied this in the setting of a Banach space, we shall confine our discussion to a complex Banach algebra A with unit 1.

Let A be a complex Banach algebra with unit 1. Let $Inv(A)$ denote the set of all invertible elements in A. Then, for each $a \in A$,

$$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin Inv(A)\}$$

Note that the set $Inv(A)$ has the following properties.

1. $1 \in Inv(A)$.
2. $0 \notin Inv(A)$.
3. If $a \in Inv(A)$ and $\lambda \neq 0$, then $\lambda a \in Inv(A)$.
4. $Inv(A)$ is an open subset of A.

One way of generalizing the idea of the spectrum is to replace the set $Inv(A)$ by some other set preferably having some of these properties. Thus the *Exponential spectrum* arises in this way. In place of $Inv(A)$ we consider

$$\exp(A) := \{\exp(a) : a \in A\}$$

Then for $a \in A$, the exponential spectrum $\sigma_{exp}(a)$ is defined by

$$\sigma_{exp}(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \exp(A)\}$$

Note that in general

$$\sigma(a) \subseteq \sigma_{exp}(a).$$

Definition 3.1 **Ransford set** Let A be a complex Banach algebra with unit 1.

An open subset Ω of A satisfying the following properties is called a *Ransford set*.

1. $1 \in \Omega$.
2. $0 \notin \Omega$.

3. If $a \in \Omega$ and $\lambda \neq 0$, then $\lambda a \in \Omega$.

Note that $Inv(A)$ and $\exp(A)$ are Ransford sets.

Definition 3.2 **Ransford spectrum** let $a \in A$ and Ω be a Ransford set. Then the *Ransford spectrum of a with respect to the Ransford set* Ω is defined as follows:

$$\sigma^{\Omega}(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \Omega\}.$$

Note that $Inv(A)$ is a Ransford set and the usual spectrum $\sigma(a)$ is nothing but $\sigma^{Inv(A)}(a)$, that is, Ransford spectrum with respect to the Ransford set $Inv(A)$, in this notation. Similar comments hold about $\exp(A)$ and exponential spectrum.

For this spectrum, Ransford proved the following properties.

Theorem 3.3

1. $\sigma^{\Omega}(0) = \{0\}$ *and* $\sigma^{\Omega}(1) = \{1\}$
2. *If for* $a \in A$, $\sigma^{\Omega}(a) \neq \emptyset$, *then* $\sigma^{\Omega}(a)$ *is compact*
3. *Let* $E := \{a \in A : \sigma^{\Omega}(a) \neq \emptyset\}$. *Then the map* $a \to \sigma^{\Omega}(a)$ *is an upper semicontinuous function from* E *to compact subsets of* $\mathbb{C}$.

A proof of this theorem as well as several properties of Ransford spectrum can be found in Ransford's article cited above. Subsequent studies of Ransford spectrum can be found in [2, 17].

4 Numerical range

Definition 4.1 *Numerical Range* Let A be a Banach algebra and $a \in A$. The *numerical range* of a is defined by

$$V(a) := \{f(a) : f \in A', f(1) = 1 = \|f\|\},$$

where A' denotes the dual space of A.

The *numerical radius* $v(a)$ is defined as

$$v(a) := \sup\{|\lambda| : \lambda \in V(a)\}$$

Let A be a Banach algebra and $a \in A$. Then a is said to be *Hermitian* if $V(a) \subseteq \mathbb{R}$.

Definition 4.2 **Spatial Numerical Range**

Let X be a Banach space and $T \in B(X)$. Let X' denote the dual space of X. The *spatial numerical range* of T is defined by

$$W(T) = \{f(Tx) : f \in X', \|f\| = f(x) = 1 = \|x\|\}.$$

For an operator T on a Banach space X, the spatial numerical range $W(T)$ and the numerical range $V(T)$, where T is regarded as an element of the Banach algebra $B(X)$, are related by the following:

$$\overline{\mathrm{Co}}\, W(T) = V(T)$$

where $\overline{\mathrm{Co}}\, E$ denotes the closure of the convex hull of $E \subseteq \mathbb{C}$.

The following theorem gives the relation between the spectrum and numerical range.

Theorem 4.3 *Let A be a Banach algebra and $a \in A$.*
Then the numerical range $V(a)$ is a closed convex set containing $\sigma(a)$. Thus $\overline{Co}(\sigma(a)) \subseteq V(a)$. Hence
$r(a) \leq v(a) \leq \|a\| \leq ev(a)$.

A proof of this can be found in [3].

5 Condition spectrum

Next we discuss one more such extension in terms of the condition number.

Definition 5.1 **Condition Number** Let A be a complex Banach algebra with unit 1. The *condition number* of an invertible element $a \in A$ is defined as $\|a\|\|a^{-1}\|$ and denoted by $\kappa(a)$. It is convenient to make a convention that $\kappa(a) = \infty$ if a is not invertible.

We shall use this convention through out. The condition number is a very useful concept and arises naturally in solving a system of equations. Specifically it is a measure of the sensitivity of the answer to a problem to small changes in the initial data of the problem.

For a fixed $0 < \epsilon < 1$, define

$$\Omega_\epsilon := \left\{ a \in Inv(A) : \kappa(a) < \frac{1}{\epsilon} \right\}.$$

As 0 is not invertible, $0 \notin \Omega_\epsilon$, also $1 \in \Omega_\epsilon$, since $\|1\|\left\|1^{-1}\right\|$=1. Note that

$$\|a\|\left\|a^{-1}\right\| = \|za\|\left\|(za)^{-1}\right\|, \forall z \in \mathbb{C} \setminus \{0\}$$

and this proves that if $a \in \Omega_\epsilon$ and $\lambda \neq 0$, then $\lambda a \in \Omega_\epsilon$. The map $a \to \|a\|\left\|a^{-1}\right\|$ is continuous and hence Ω_ϵ is an open set. These observations prove that Ω_ϵ is a Ransford set.

Definition 5.2 **Condition spectrum**
Let $0 < \epsilon < 1$. The *ϵ-condition spectrum* of a for this ϵ is defined by

$$\sigma_\epsilon(a) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \Omega_\epsilon\}$$
$$= \left\{\lambda \in \mathbb{C} : \kappa(\lambda 1 - a) \geq \frac{1}{\epsilon}\right\}$$

with the convention that $\kappa(\lambda 1 - a) = \infty$ when $\lambda - a$ is not invertible. *Condition spectral radius* $r_\epsilon(a)$ is defined by

$$r_\epsilon(a) := \sup\{|z| : z \in \sigma_\epsilon(a)\}.$$

This condition spectrum was defined for the first time in [18]. Suppose X is a Banach space, $T : X \to X$ is a bounded linear map and $y \in X$. Consider the operator equation

$$Tx - \lambda x = y$$

Then

- $\lambda \notin \sigma(T)$ implies that this operator equation is solvable
- $\lambda \notin \sigma_\epsilon(T)$ implies that this operator equation has a stable solution.

In view of this, the ϵ-condition spectrum is expected to be a useful tool in numerical solutions of operator equations.

Next we give a few elementary properties of the condition spectrum.

Theorem 5.3 *Let A be a complex Banach algebra with unit* 1.

1. $\sigma_\epsilon(0) = \{0\}$ *and* $\sigma_\epsilon(1) = \{1\}$.
2. *If* $0 < \epsilon_1 < \epsilon_2 < 1$, *then* $\sigma_{\epsilon_1}(a) \subseteq \sigma_{\epsilon_2}(a)$ *for every* $a \in A$
3. $\sigma(a) \subseteq \sigma_\epsilon(a)$ *for every* $a \in A$. *In fact*
$$\sigma(a) = \bigcap_{0<\epsilon<1} \sigma_\epsilon(a)$$
4. $\sigma_\epsilon(a)$ *is a non empty compact subset of* $\mathbb{C}$ *for every* $a \in A$
5. *The map* $a \to \sigma_\epsilon(a)$ *is an upper semi continuous function from* A *to compact subsets of* $\mathbb{C}$.

A proof can be found in [18]

Next, we shall see a few examples.

Example 5.4 **Diagonal matrix**

Let $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and let $M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then

$$\|M-\lambda I\| = \max\{|\lambda-\lambda_1|, |\lambda-\lambda_2|\}$$
$$\|(M-\lambda I)^{-1}\| = \max\left\{\frac{1}{|\lambda-\lambda_1|}, \frac{1}{|\lambda-\lambda_2|}\right\}.$$

Hence

$$\sigma_\epsilon(M) = \left\{\lambda : \frac{|\lambda-\lambda_1|}{|\lambda-\lambda_2|} \geq \frac{1}{\epsilon}\right\} \cup \left\{\lambda : \frac{|\lambda-\lambda_2|}{|\lambda-\lambda_1|} \geq \frac{1}{\epsilon}\right\}.$$

Example 5.5 **Triangular matrix**

Let $R : \mathbb{C}^2 \to \mathbb{C}^2$ defined as $R(x, y) = (0, x)$ for all (x, y) in $\mathbb{C}^2$ (truncation of right shift operator). Considering R as an operator on $\mathbb{C}^2$ we get

$$\|R-\lambda I\|_1 = \|R-\lambda\|_\infty = 1 + |\lambda|$$
$$\|(R-\lambda I)^{-1}\|_1 = \|(R-\lambda)^{-1}\|_\infty = \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}.$$

Hence in both $(\mathbb{C}^2, \|\ \|_1)$ and $(\mathbb{C}^2, \|\ \|_\infty)$

$$\sigma_\epsilon(R) = \left\{\lambda : |\lambda| \leq \frac{\sqrt{\epsilon}}{1-\sqrt{\epsilon}}\right\}.$$

Example 5.6 **Right shift operator**

Let R be the right shift operator on ℓ^p where $p = 1$ or ∞. We can show

$$\|R-\lambda I\|_1 = \|R-\lambda I\|_\infty = |\lambda| + 1.$$

For $|\lambda| > 1$ $(R-\lambda I)^{-1}$ exists and

$$\|(R-\lambda I)^{-1}\|_1 = \|(R-\lambda I)^{-1}\|_\infty = \frac{1}{|\lambda|-1}.$$

It is known that $\sigma(R) = \{\lambda : |\lambda| \leq 1\}$ Also we can show

$$\sigma_\epsilon(R) = \left\{\lambda : |\lambda| \leq \frac{1+\epsilon}{1-\epsilon}\right\}.$$

In the next theorem, we list some more properties of the condition spectrum.

Theorem 5.7 *Let A be a complex unital Banach algebra, $a \in A$ and $0 < \epsilon < 1$.*

1. *Suppose $a \neq \lambda$ for every $\lambda \in \mathbb{C}$. Then $\sigma_\epsilon(a)$ has no isolated points.*
2. *If a is not a scalar multiple of the identity, then for each $\lambda_0 \in \sigma(a)$, there exist $r > 0$ such that $D(\lambda_0, r) \subseteq \sigma_\epsilon(a)$. In particular, $\sigma(a) \subsetneqq \sigma_\epsilon(a)$.*
3. *If $\sigma_\epsilon(a) = \sigma(a)$ then $a = \lambda_0$ for some $\lambda_0 \in \mathbb{C}$.*

4. *$\sigma_\epsilon(a)$ has a finite number of components and every component of $\sigma_\epsilon(a)$ contains an element from $\sigma(a)$.*
5. *If $M \in \mathbb{C}^{n\times n}$ and $\sigma_\epsilon(M)$ has n components, then M is diagonalizable.*

For a proof, see [18]

Corollary 5.8 *If $\sigma_\epsilon(a) = \{\lambda_0\}$ for some $\lambda_0 \in \mathbb{C}$, then $a = \lambda_0$.*

Remark 5.9 A very well known classical problem in operator theory known as "$T = I$?" *problem*, asks the following question: *Let T be an operator on a Banach space. Suppose $\sigma(T) = \{1\}$. Under what additional conditions can we conclude $T = I$?*

From the above corollary it follows that if $\sigma_\epsilon(T) = \{1\}$ then $T = I$. In other words if we replace spectrum by ϵ-condition spectrum in the "$T = I''$ problem, then no additional conditions are required.

Remark 5.10 Numerical Range and condition spectrum are related as follows:

Let A be a complex unital Banach algebra. Let $a \in A$.

If $\lambda \in \sigma_\epsilon(a)$, then we can prove that

$$d(\lambda, V(a)) \le \epsilon\|\lambda - a\|$$

A proof can be found in [18].

6 Pseudospectrum

We now discuss yet another important and popular set related to the spectrum, namely pseudospectrum. We begin with its definition.

Definition 6.1 **Pseudospectrum** Let A be a complex Banach algebra, $a \in A$ and $\epsilon > 0$. The ϵ-pseudospectrum $\Lambda_\epsilon(a)$ of a is defined by

$$\Lambda_\epsilon(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \ge \epsilon^{-1}\}$$

with the convention that $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

This definition and many results in this section can be found in [14]. The book [32] is a standard reference on Pseudospectrum. It contains a good amount of information about the idea of pseudospectrum, (especially in the context of matrices and operators), historical remarks and applications to various fields. Another useful source is the website [33].

Remark 6.2 **Other definitions**

Some authors, in particular, Trefethen, have defined the following set as the ϵ-pseudospectrum of a:

$$\Lambda_\epsilon^*(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| > \epsilon^{-1}\}.$$

There are some significant changes in these two definitions.

1. $\Lambda_\epsilon(a)$ is a compact subset of $\mathbb{C}$ whereas $\Lambda^*_\epsilon(a)$ is not.
2. The map $\epsilon \mapsto \Lambda_\epsilon(a)$ is right continuous but the map $\epsilon \mapsto \Lambda^*_\epsilon(a)$ is not.

In the case of most of the other results about $\Lambda_\epsilon(a)$, our methods can be easily modified to obtain analogous results for $\Lambda^*_\epsilon(a)$. In general, $\Lambda_\epsilon(a)$ is not the closure of $\Lambda^*_\epsilon(a)$.

However, this is true in many cases. We shall see some information about this later.

One reason given by some authors for accepting $\Lambda^*_\epsilon(a)$ as the definition of pseudospectrum is that if T is a bounded operator on a Banach space, then

$$\Lambda^*_\epsilon(T) = \bigcup_{\|S\|<\epsilon} \sigma(T+S).$$

However, this is not the case for an arbitrary element of a Banach algebra (We shall see such an example).

A more detailed discussion on these two ways of defining pseudospectrum can be found in [31]

The following theorem gives some elementary properties of the pseudospectrum.

Theorem 6.3 *Let A be a complex Banach algebra. Then*

1. $\Lambda_\epsilon(a)$ *is a non-empty compact subset of* $\mathbb{C}$ $(a \in A, \epsilon > 0)$.
2. $\sigma(a) = \bigcap_{\epsilon>0} \Lambda_\epsilon(a)$ $(a \in A)$.
3. $\Lambda_{\epsilon_1}(a) \subset \Lambda_{\epsilon_2}(a)$ $(a \in A, 0 < \epsilon_1 < \epsilon_2)$.
4. $\Lambda_\epsilon(a+\lambda) = \lambda + \Lambda_\epsilon(a)$ $(\lambda \in \mathbb{C})$.
5. $\Lambda_\epsilon(\lambda a) = \lambda \Lambda_{\frac{\epsilon}{|\lambda|}}(a)$ $(a \in A, \lambda \in \mathbb{C} \setminus \{0\}, \epsilon > 0)$.
6. $\Lambda_\epsilon(a) \subseteq D(0; \|a\| + \epsilon)$ $(a \in A, \epsilon > 0)$.
7. $\Lambda_\epsilon(a+b) \subseteq \Lambda_{\epsilon+\|b\|}(a)$ $(a, b \in A, \epsilon > 0)$.
8. $\sigma(a+b) \subseteq \Lambda_\epsilon(a)$ $(a, b \in A, \epsilon > 0, \|b\| \leq \epsilon)$, *i.e.* $\bigcup_{\|b\|\leq\epsilon} \sigma(a+b) \subseteq \Lambda_\epsilon(a)$.
9. $\Lambda_\epsilon(a) + D(0; \delta) \subseteq \Lambda_{\epsilon+\delta}(a)$.

A proof can be found in [14]

The inclusion in (8) of the above Theorem can be proper. Consider the following example:

Example 6.4 Let $A = \left\{ a \in \mathbb{C}^{2\times 2} : a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \right\}$ with norm given by $\|a\| = |\alpha| + |\beta|$. Then A is a Banach algebra. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then it can be verified that

$$\bigcup_{\|b\| \leq 1} \sigma(a+b) = D(0;1)$$

which is properly contained in

$$\Lambda_1(a) = \{\lambda \in \mathbb{C} : |\lambda|(|\lambda| - 1) \leq 1\} = D\left(0; \left(\frac{1+\sqrt{5}}{2}\right)\right).$$

Next, we consider the question of reverse inclusion in (8) of the above Theorem.

Lemma 6.5 *Suppose A is a complex Banach algebra with the following property*:

$$\forall a \in \mathrm{Inv}(A),\ \exists b \in \mathrm{Sing}(A) \text{ such that } \|a-b\| = \frac{1}{\|a^{-1}\|}. \tag{1}$$

Then $\forall a \in A$ and $\lambda \in \Lambda_\epsilon(a)$, $\exists b \in A$ such that $\|b\| \leq \epsilon$ and $\lambda \in \sigma(a+b)$.

A proof can be found in [14]

Examples of Banach algebras that satisfy the hypothesis of the above Lemma can be found in [17]. These include the algebras $C(X)$, for a compact Hausdorff space X, and $\mathbb{C}^{n \times n}\ \forall n \in \mathbb{N}$. In fact, all C^* algebras satisfy the hypothesis as given below.

Theorem 6.6 *If A is a C^* algebra, and $a \in \mathrm{Inv}(A)$, then $\exists b \in \mathrm{Sing}(A)$ such that $\|a-b\| = \frac{1}{\|a^{-1}\|}$.*

Next, we consider an example of a Banach algebra in which this condition does not hold.

Example 6.7 Consider A as in the Example 6.4 above. Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then we claim that

$$b \in A,\ \|a-b\| = \frac{1}{\|a^{-1}\|} \Rightarrow b \in \mathrm{Inv}(A).$$

For the given a, $a^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\|a^{-1}\| = 2$. Any $b \in A$ is of the form $\begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}$ and b is invertible iff $\alpha \neq 0$. Then $\|a-b\| = |1-\alpha| + |1-\beta|$. If $\|a-b\| = \frac{1}{\|a^{-1}\|}$, i.e., $|1-\alpha| + |1-\beta| = \frac{1}{2}$, then $\alpha \neq 0$. Hence b is invertible.

Corollary 6.8 *Let A be a complex Banach algebra satisfying the hypothesis of Lemma 6.5 and $a \in A$. Then*

$$\lambda \in \Lambda_\epsilon(a) \Leftrightarrow \exists b \in A \text{ with } \|b\| \leq \epsilon \text{ such that } \lambda \in \sigma(a+b).$$

Thus

$$\Lambda_\epsilon(a) = \bigcup_{\|b\| \leq \epsilon} \sigma(a+b).$$

The following theorems establish the relationships between the spectrum, the ϵ-pseudospectrum and the numerical range of an element of a Banach algebra.

Theorem 6.9 *Let A be a Banach algebra, $a \in A$ and $\epsilon > 0$. Then*

$$d(\lambda, V(a)) \leq \frac{1}{\|(\lambda - a)^{-1}\|} \leq d(\lambda, \sigma(a)) \quad \forall \lambda \in \mathbb{C} \setminus \sigma(a). \tag{2}$$

Thus

$$\sigma(a) + D(0;\epsilon) \subseteq \Lambda_\epsilon(a) \subseteq V(a) + D(0;\epsilon). \tag{3}$$

Next we consider the question of equality in some of these inclusions.

Definition 6.10 Let A be a Banach algebra and $a \in A$. We define a to be of G_1*-class* if

$$\|(z-a)^{-1}\| = \frac{1}{\mathrm{d}(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a). \tag{4}$$

The following lemma is elementary.

Lemma 6.11 *Let A be a Banach algebra and $a \in A$. Then*

$$\Lambda_\epsilon(a) = \sigma(a) + D(0;\epsilon) \quad \forall \epsilon > 0 \tag{5}$$

iff a is of G_1-class.

Remark 6.12 The idea of G_1-class is due to Putnam who defined it for operators on Hilbert spaces. (See [26, 27].) It is known that the G_1-class properly contains the class of seminormal operators ($TT^* \leq T^*T$ or $T^*T \leq TT^*$) and this class properly contains the class of normal operators. Using Gelfand- Naimark theorem, we can make similar statements about elements in a C^* algebra.

In the finite dimensional case, G_1 operators are normal.

Also it is easy to see that every element in a uniform algebra is of G_1-class.

In particular, normal elements are hyponormal. In general, the equation (5) may hold, for every $\epsilon > 0$, for an element of a C^*-algebra even though it is not normal.

Consider the right shift operator R on $\ell^2(\mathbb{N})$. It is not normal but $\Lambda_\epsilon(R) = \sigma(R) + D(0;\epsilon) = D(0; 1+\epsilon)\, \forall \epsilon > 0$. R is, however, a hyponormal operator.

The following theorem shows that the numerical range $V(a)$ of a is determined by certain closed half-planes related to the pseudospectrum $\Lambda_\epsilon(a)$.

Theorem 6.13 *Let A be a Banach algebra, $a \in A$ and $\epsilon > 0$. If H is a closed half-plane in $\mathbb{C}$ such that*

$$\Lambda_\epsilon(a) \subseteq H + D(0;\epsilon) \quad \forall \epsilon > 0. \tag{6}$$

Then $V(a) \subseteq H$.

A proof can be found in [14].

The following corollary gives an equivalent condition in terms of the ϵ-pseudospectrum for an element of a Banach algebra to be Hermitian.

Corollary 6.14 *Let A be a Banach algebra and $a \in A$. Then a is Hermitian iff*

$$\Lambda_\epsilon(a) \subseteq \{z \in \mathbb{C} : |\mathrm{Im}\ z| \le \epsilon\} \quad \forall \epsilon > 0. \tag{7}$$

The numerical range of an element of a Banach algebra is a compact convex subset of $\mathbb{C}$ containing its spectrum, and hence it also contains the closure of the convex hull of the spectrum. In some cases, as given below, the equality holds.

Corollary 6.15 *Let A be a Banach algebra and $a \in A$. Suppose a is of G_1-class. Then $V(a) = \mathrm{Co}\ \sigma(a)$ and $\|a\| \le e\, r(a)$.*

Following is an interesting theorem that is a consequence of the above considerations.

Theorem 6.16 *Let A be a Banach algebra. Suppose a is of G_1-class for every $a \in A$. Then A is commutative, semisimple and hence isomorphic and homeomorphic to a function algebra.*

Proof By the above result, $\|a\| \le er(a)\, \forall a \in A$. Hence A is commutative by a theorem of Hirschfeld and Zelazko [3]. Also, the condition $\|a\| \le er(a)\, \forall a \in A$ implies that A is semisimple. □

Next two propositions give relationship between condition spectrum and pseudospectrum of an element in a complex unital Banach algebra. Their proofs are elementary and can be found in [19].

Proposition 6.17 *Let A be a complex Banach algebra with unit 1, $a \in A$ and $0 < \epsilon < 1$. Then $\sigma_\epsilon(a) \subseteq \Lambda_{\frac{2\epsilon\|a\|}{1-\epsilon}}(a)$.*

Proposition 6.18 *Let A be a complex Banach algebra with unit 1 and $\epsilon > 0$. Suppose $a \in A$ is not a scalar multiple of 1 and let*
$M := \inf\{\|\lambda - a\| : \quad \lambda \in \mathbb{C}\}$*. Then $\Lambda_\epsilon(a) \subseteq \sigma_{\frac{\epsilon}{M}}(a)$.*

Remark 6.19 If $a = \mu.1$ for some $\mu \in \mathbb{C}$, then ϵ-condition spectrum of a is the singleton set $\{\mu\}$ and ϵ-pseudospectrum is the closed ball with centre μ and radius ϵ. Thus the condition on a can not be dropped from the above proposition.

The following theorem involves the analytical functional calculus for elements of a Banach algebra.

Theorem 6.20 *Let A be a Banach algebra and $a \in A$. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\Lambda_\epsilon(a)$ and Γ be a contour that surrounds $\Lambda_\epsilon(a)$ in Ω. Let f be analytic in Ω. We recall the definition of $\tilde{f}(a)$ in the analytical functional calculus as*

$$\tilde{f}(a) = \frac{1}{2\pi i} \int_\Gamma (z-a)^{-1} f(z) dz \tag{8}$$

Then

$$\|\tilde{f}(a)\| \leq \frac{Ml}{2\pi\epsilon} \tag{9}$$

where $l =$ length of Γ and $M = \sup\{|f(z)| : z \in \Gamma\}$.

The following corollary gives an equivalent condition in terms of the ϵ-pseudospectrum for an element of a Banach algebra to be a scalar (i.e. a scalar multiple of the identity).

Corollary 6.21 *Let A be a Banach algebra, $a \in A$ and $\mu \in \mathbb{C}$. Then*

$$a = \mu \Leftrightarrow \Lambda_\epsilon(a) = D(\mu, \epsilon) \quad \forall \epsilon > 0.$$

Proof If $a = \mu$, it is trivial to see that $\Lambda_\epsilon(A) = D(\mu, \epsilon) \quad \forall \epsilon > 0$. For the converse part, by (4) of Theorem 6.3, we may assume that $\mu = 0$. Let $f(z) = z$ and $\Gamma = \{z \in \mathbb{C} : |z| = \epsilon\}$. Then, with the notations of the above Theorem , $M = \epsilon$ and $l = 2\pi\epsilon$. Hence by the above Theorem , $\|a\| \leq \epsilon$. Since this is true $\forall \epsilon > 0$, $a = 0 = \mu$. □

The following corollary gives an equivalent condition in terms of the ϵ-pseudospectrum for an element of a Banach algebra to be a Hermitian idempotent.

Corollary 6.22 *Let A be a Banach algebra and $a \in A$. Then*

$$\Lambda_\epsilon(a) = D(0; \epsilon) \cup D(1; \epsilon) \quad \forall \epsilon > 0 \tag{10}$$

if and only if a is a non-trivial(that is, different from 0 and 1) Hermitian idempotent and $\|a\| = 1$.

Next we consider some topological properties of the ϵ-pseudospectrum of an element of a Banach algebra, namely that the ϵ-pseudospectrum has no isolated points, and that it has a finite number of components.

Theorem 6.23 *Let A be a Banach algebra, $a \in A$ and $\epsilon > 0$. Then the ϵ-pseudospectrum $\Lambda_\epsilon(a)$ of a has no isolated points. Also the ϵ-pseudospectrum $\Lambda_\epsilon(a)$ of a has a finite number of components and each component of $\Lambda_\epsilon(a)$ contains an element of $\sigma(a)$.*

See [14] for a proof.

The above Theorem helps to determine certain properties of a matrix when its ϵ-pseudospectrum is known.

Corollary 6.24 *Let $M \in \mathbb{C}^{n \times n}$ and $\epsilon > 0$.*

1. *If $\Lambda_\epsilon(M)$ has n components, then M is diagonalizable.*
2. *If each of these components is a disc of radius ϵ and $\|\cdot\| = \|\cdot\|_2$ then M is normal.*
3. *If $\|\cdot\| = \|\cdot\|_2$, then $\Lambda_\epsilon(M) = D(\mu;\epsilon)$ iff $M = \mu I$.*
4. *If $\|\cdot\| = \|\cdot\|_2$, then $\Lambda_\epsilon(M) = D(0;\epsilon) \cup D(1;\epsilon)$ iff M is a non-trivial orthogonal projection.*

7 (n,ϵ)-pseudospectrum

In this section, we discuss one more set related to the spectrum that is very important from the point of view of approximation of the spectrum. It is defined as follows.

Definition 7.1 Let A be a unital Banach algebra, $a \in A$, $\epsilon > 0$ and n a nonnegative integer. The (n,ϵ)-pseudospectrum of a is defined by

$$\Lambda_{n,\epsilon}(a) := \sigma(a) \cup \left\{ \lambda \notin \sigma(a) : \|(\lambda - a)^{-2^n}\|^{1/2^n} \geq \frac{1}{\epsilon} \right\}.$$

This set was first introduced by Hansen [11, 12] for the operators on a Hilbert space. This idea was extended to cover the operators on Banach spaces by Seidel[29]. It was further extended for an element in a Banach algebra and more investigations were carried out in [6, 7].

The following functions γ_n are quite useful in describing and proving properties of $(n,\epsilon) - pseudospectrum$.

Definition 7.2 Let A be a unital Banach algebra, $a \in A$, $\epsilon > 0$, $z \in \mathbb{C}$ and n a nonnegative integer. The functions γ_n and γ are defined as follows:

$$\gamma_n(a,z) := \|(z - a)^{-2^n}\|^{-1/2^n} \quad \text{if} \quad z \notin \sigma(a)$$

and $= 0$ if $z \in \sigma(a)$.

$$\gamma(a,z) := d(z, \sigma(a)).$$

First note that

$$\Lambda_{n,\epsilon}(a) := \{\lambda \in \mathbb{C} : \gamma_n(a,\lambda) \leq \epsilon\}.$$

We observe that the $(0,\epsilon)$-pseudospectrum is nothing but the usual ϵ-pseudospectrum. Also, for a normal element in a C^* algebra, we have $\Lambda_{n,\epsilon}(a) = \Lambda_\epsilon(a) = \sigma(a) + D(0,\epsilon)$ for all n.

The following theorem provides some elementary properties of the (n, ϵ)-pseudospectrum. Its proof is given in [6].

Theorem 7.3 *Let A be a Banach algebra, $a, b \in A$, n a nonnegative integer and $\epsilon > 0$. Then the following statements hold:*

1. $\Lambda_{n,\epsilon}(\lambda) = D(\lambda, \epsilon)\ \forall \lambda \in \mathbb{C}$.
2. $\Lambda_{n+1,\epsilon}(a) \subseteq \Lambda_{n,\epsilon}(a)$.
3. $\sigma(a) = \bigcap_{\epsilon > 0} \Lambda_{n,\epsilon}(a)$.
4. $\Lambda_{n,\epsilon_1}(a) \subseteq \Lambda_{n,\epsilon_2}(a)$ for $0 < \epsilon_1 < \epsilon_2$.
5. $\Lambda_{n,\epsilon}(a + \lambda) = \lambda + \Lambda_{n,\epsilon}(a)$ for $\lambda \in \mathbb{C}$.
6. $\Lambda_{n,\epsilon}(\lambda a) = \lambda \Lambda_{n,\frac{\epsilon}{|\lambda|}}(a)$ for $\lambda \in \mathbb{C} \setminus \{0\}$.
7. $\Lambda_{n,\epsilon}(a) \subseteq D(0, \|a\| + \epsilon)$.
   Further, if a is invertible and $0 < \epsilon < \frac{1}{\|a^{-1}\|}$, then
   $$\Lambda_{n,\epsilon}(a) \subseteq \left\{ z \in \mathbb{C} : \frac{1}{\|a^{-1}\|} - \epsilon \leq |z| \leq \|a\| + \epsilon \right\}.$$
8. *$\Lambda_{n,\epsilon}(a)$ is a non-empty compact subset of $\mathbb{C}$.*

The following theorem says that $\Lambda_{n,\epsilon}(a)$ is an approximation of an ϵ-neighborhood of the $\sigma(a)$ for large values of n. Thus if we have a good method of computing $\Lambda_{n,\epsilon}(a)$, then we can get information about $\sigma(a)$. This aspect of computing (n, ϵ)-pseudospectrum is discussed by Hansen for bounded operators on a separable Hilbert space. This involves the use of the functions γ_n.

Theorem 7.4 *Let A be a Banach algebra, $a \in A$ and $\epsilon > 0$. Then*

$$\sigma(a) + D(0, \epsilon) = \bigcap_{n \in \mathbb{Z}_+} \Lambda_{n,\epsilon}(a).$$

Further, $d_H(\Lambda_{n,\epsilon}(a), \sigma(a) + D(0; \epsilon)) \to 0$ as $n \to \infty$.

A proof can be found in [6]. The following theorem gives another way of looking at this approximation.

Theorem 7.5 *Let A be a Banach algebra, $a \in A$. Then for $0 < \epsilon < \eta$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\sigma(a) + D(0, \epsilon) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \sigma(a) + D(0, \eta)$$

A prooof can be found in [6].

The inclusion $\sigma(a) + D(0, \epsilon) \subseteq \Lambda_{n,\epsilon}(a)$ can be proper. We give an example below.

Example 7.6 Let the Banach algebra A and an element $a \in A$ be as in the Example 6.4. Then it can be shown that there exists $\lambda \in \Lambda_{n,\epsilon}(a)$ but $\lambda \notin \sigma(a) + D(0, \epsilon)$. See [6] for details.

We now introduce a class of elements that have some special properties with respect to the spectrum and (n, ϵ)-pseudospectrum.

Definition 7.7 Let A be a unital Banach algebra and n a non-negative integer. An element $a \in A$ is said to be of G_n -class if

$$\|(\lambda - a)^{-2^{n-1}}\|^{1/2^{n-1}} = \frac{1}{d(\lambda, \sigma(a))} \ \forall \lambda \notin \sigma(a).$$

This means $\gamma_{n-1}(a, \lambda) = d(\lambda, \sigma(a) =: \gamma(\lambda, a)$ for all $\lambda \in \mathbb{C}$.

It follows from the definition that a is of G_n -class iff $\Lambda_{n-1,\epsilon}(a) = \sigma(a) + D(0, \epsilon) \, \forall \epsilon > 0$. For $n = 1$, the above definition coincides with the familiar definition of G_1-class .

In the algebra $\mathcal{A}$ of the above Example 6.4, $b = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in G_n$ iff $y = 0$.

Thus, in this algebra, G_1 and G_n -class elements are the same for all n.

Note that

$$\gamma_{n-1}(a, z) \leq \gamma_n(a, z) \leq d(z, \sigma(a)) \quad \text{for all} \quad z \in \mathbb{C}.$$

Hence G_n -class is contained in G_{n+1} -class . Thus we find that if a is of G_n -class, then $\forall m \geq n$, $\Lambda_{m,\epsilon}(a) = \sigma(a) + D(0, \epsilon) \, \forall \epsilon > 0$.

The inclusion $G_n \subseteq G_{n+1}$ can be proper. An example to support this claim can be found in [6].

The following theorem gives a characterization of scalar elements in a Banach algebra in terms of its (n, ϵ)-pseudospectrum.

Theorem 7.8 *Suppose A is a Banach algebra, $a \in A$ and n a non-negative integer. Then*

$$a = \lambda \Longleftrightarrow \Lambda_{n,\epsilon}(a) = D(\lambda, \epsilon) \ \forall \epsilon > 0.$$

A proof can be found in [6].

The next theorem gives some topological properties of (n, ϵ)-pseudospectrum. Note that these are very similar to the corresponding properties of the pseudospectrum.

Theorem 7.9 *Let A be a Banach algebra, $a \in A$, $n \in \mathbb{Z}_+$ and $\epsilon > 0$. Then*

1. *$\Lambda_{n,\epsilon}(a)$ has no isolated points*
2. *$\Lambda_{n,\epsilon}(a)$ has a finite no of components and each component contains at least one element of $\sigma(a)$.*

A proof is given in [6].

8 Stability of the spectrum and related sets

As we have already observed, the spectrum is a very useful concept in several applications. In concrete cases, it has well known interpretations such as the spectrum of a square matrix or spectrum of an operator. Thus the computation of the spectrum of an element is an important task. On the other hand, as observed in the Introduction, it is well known that the map $a \mapsto \sigma(a)$ is not continuous in general. There are many examples in the literature to demonstrate this. In particular, if T is a bounded linear operator defined on a separable Hilbert space H with an orthonormal basis $\{e_j\}$, then the spectrum $\sigma(T)$ of T does not depend continuously on the matrix entries $\langle Te_j, e_i\rangle$. We may consider the following example given in [12].

Example 8.1 Let δ be a real number and let $T_\delta : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be defined by $(T_\delta x)(n) = \delta x(n+1)$ if $n = 0$ and $(T_\delta x)(n) = x(n+1)$ if $n \neq 0$ for $x = \{x(n)\} \in \ell^2(\mathbb{Z})$. Then it can be shown that for each $\delta \neq 0$, the spectrum $\sigma(T_\delta)$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ but for $\delta = 0$, we have $\sigma(T_0) = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disc. On the other hand $T_\delta \to T_0$ as $\delta \to 0$.

This situation is of concern to a numerical analyst because if one does computation of the spectrum of T_0 on a computer, then due to round off and truncation errors, one gets the solution of a slightly perturbed problem, that is the spectrum of T_δ for a small value of δ. But as the above example shows, this solution will be quite away from the desired solution.

We discuss the stability of the other spectrum related sets beginning with the Numerical range. Let us recall the following notation.

Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $K(\mathbb{C})$ denote the set of compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric defined as

$$d_H(\Lambda, \Delta) = max\left\{\sup_{s\in\Lambda} d(s, \Delta), \sup_{t\in\Delta} d(t, \Lambda)\right\}.$$

Theorem 8.2 *Let A be a complex unital Banach algebra and $a, b \in A$. Then $d_H(V(a), V(b)) \leq \|a - b\|$. Thus the map $a \mapsto V(a)$ is continuous, in fact, uniformly continuous.*

Proof Let $s \in V(a)$. Then $s = f(a)$ for some $f \in A'$ with $\|f\| = 1 = f(1)$. Then $f(b) \in V(b)$. Hence $d(s, V(b)) \leq |s - f(b)| = |f(a) - f(b)| \leq \|a - b\|$. Similarly, for every $t \in V(b)$, we can show that $d(t, V(a)) \leq \|a - b\|$. $\square$

Next we consider the question of stability of ϵ-pseudospectrum and (n, ϵ)-pseudospectrum. Since ϵ-pseudospectrum is a special case of (n, ϵ)-pseudospectrum, we shall only consider the results about (n, ϵ)-pseudospectrum. We begin with the follwing important theorem.

Theorem 8.3 *Let A be a complex unital Banach algebra. Then for a fixed element $a \in A$, the map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is right continuous.*

This is proved in [6]. This theorem says that the map is continuous whenever it is left continuous. In the following theorem, we study some equivalent conditions for (left) disconinuity of the map.

Theorem 8.4 *Let A be a complex unital Banach algebra, $a \in A$, $\epsilon_0 > 0$ and n a non-negative integer. Then the following statements are equivalent.*

1. *The map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is left discontinuos at ϵ_0.*
2. *The level set $\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) = \epsilon_0\}$ contains a non-empty open set.*
3. *The closure of the set $\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) < \epsilon_0\}$ is properly contained in the set $\{\lambda \in \mathbb{C} : \gamma_n(a, \lambda) \leq \epsilon_0\}$.*

This theorem is proved in [7]. The following theorem gives some equivalent conditions for the contnuity of this map.

Theorem 8.5 *Let A be a complex unital Banach algebra, $a_0 \in A$, $\epsilon_0 > 0$ and n a non-negative integer. Then the following statements are equivalent.*

1. *The map $\epsilon \mapsto \Lambda_{n,\epsilon}(a)$ is continuos at ϵ_0.*
2. *The map $a \mapsto \Lambda_{n,\epsilon}(a)$ is continuos at a_0.*
3. *The map $(\epsilon, a) \mapsto \Lambda_{n,\epsilon}(a)$ is continuos at (ϵ_0, a_0) with respect to the metric in the domain given by*

$$\|(\epsilon_1, a_1) - (\epsilon_2, a_2)\| = |\epsilon_1 - \epsilon_2| + \|a_1 - a_2\|$$

   for all positive ϵ_1, ϵ_2 and $a_1, a_2 \in A$ and the Hausdorff metric in the codomain.
4. *The level set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) = \epsilon_0\}$ does not contain any non-empty open set.*
5. *The closure of the set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) < \epsilon_0\}$ is equal to the set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) \leq \epsilon_0\}$.*

This is proved in [7]. Essentially this theorem says that the question of continuity of this map depends upon whether the level set $\{\lambda \in \mathbb{C} : \gamma_n(a_0, \lambda) = \epsilon_0\}$ contains any non-empty open set or not. A natural question here is what are the examples of Banach algebras and elements in those Banach algebras where this condition is satisfied. We shall need some definitions to answer that question.

Definition 8.6 A Banach space X is said to be *complex uniformly convex* if for every $\epsilon > 0$, $\exists \delta > 0$ such that

$$x, y \in X, \|y\| \geq \epsilon \text{ and } \|x + \zeta y\| \leq 1 \quad \forall \zeta \in \mathbb{C} \text{ with } |\zeta| \leq 1 \Rightarrow \|x\| \leq 1 - \delta.$$

Note that all uniformly convex spaces are complex uniformly convex. Thus Hilbert spaces and L^p spaces with $1 < p < \infty$ are complex uniformly convex. It is known that L^1 is complex uniformly convex, though not uniformly convex (See [9]). Also L^∞ is not complex uniformly convex, but its dual $(L^\infty)'$ is (See [30]). We can now answer the question raised in the last paragraph.

Theorem 8.7 *Let A be a complex unital Banach algebra, $a_0 \in A$, $\epsilon_0 > 0$ and n a non-negative integer. Then one and hence all the conditions of Theorem 8.5 are satisfied if any one of the following holds.*

1. *a_0 is of G_{n+1}-class.*
2. *The resolvent set $\mathbb{C} \setminus \sigma(a_0)$ is a connected subset of $\mathbb{C}$.*
3. *$A = B(X)$ with X or its dual X' is complex uniformly convex.*

A proof can be found in [7].

Remark 8.8 The above theroem helps in obtaining several examples of Banach algebras and elements in those Banach algebras where the condition for continuity of this map is satisfied. Let X be a Banach space and $A = B(X)$. Let K be a compact linear map on X. Then the spectrum $\sigma(K)$ of K is a countable set, hence its complement in $\mathbb{C}$ is connected. It follows by the above theorem that for any non-negative integer n and any positive ϵ, the map $T \mapsto \Lambda_{n,\epsilon}(T)$ is continuous at K. In particular, this happens when X is finite dimensional. Also the map $T \mapsto \Lambda_{n,\epsilon}(T)$ is continuous at every $T \in B(X)$, when X is a Hilbert space or $X = L^p$ with $1 \leq p \leq \infty$, because, in this case, X or its dual X' is complex uniformly convex.

Next we give an example of a bounded linear operator T on a Banach space X that does not satisfy any of the equivalent conditions given in Theorem 8.5. In particular, the map $S \mapsto \Lambda_{n,\epsilon}(S), S \in B(X)$ is not continuous at $S = T$.

Example 8.9 Let m be a non-negative integer, $X = \ell^\infty(\mathbb{Z})$ with the norm defined by

$$\|x\| := \sum_{k=0}^{m-1} |x_k| + \sup\{|x_k| : k \in \mathbb{Z} \setminus \{0, \ldots, m-1\}, \quad x \in X.$$

This norm is equivalent to the usual supnorm on X. However neither X nor its dual X' is complex uniformly convex. Let $M > 4$. Define $T : X \to X$ by $(Tx)(k) = \alpha_k x_{k+1}$ where $\alpha_k = \dfrac{1}{M}$ for $k \in \{0, \ldots, m-1\}$ and $\alpha_k = 1$ for $k \in \mathbb{Z} \setminus \{0, \ldots, m-1\}$. Then it can be shown that there exists $0 < \delta < \dfrac{1}{M}$ such that $B(0,\delta) \subseteq \mathbb{C} \setminus \sigma(T)$ and $\|(T - z)^{-m}\| = M^m$ for all $z \in B(0,\delta)$. The details of this computation can be found in [7]. This shows that the level set $\{\lambda \in \mathbb{C} : \gamma_n(T, \lambda) = 1/M\}$, where $m = 2^n$ contains a non-empty open set $B(0,\delta)$. This example was mentioned in [29].

The above example shows that, in general, the map $a \mapsto \Lambda_{n,\epsilon}(a)$ may not be continuos at a_0. However this phenomenon of discontinuity can be controlled by taking large values of n. This was first mentioned in [29]. The following theorem, proved in [7] elaborates this idea.

Theorem 8.10 *Let A be a complex unital Banach algebra, $a \in A$, $\epsilon_0 > 0$. Then the following statements hold.*

1. *For every* $\eta_1 > 0$, *there exists* $n_1 \in \mathbb{N}$ *such that* $d_H(\Lambda_{n,\epsilon}(a), \sigma(a) + D(0,\epsilon)) < \eta_1$ *for all* $n \geq n_1$ *and all* $\epsilon \leq \epsilon_0$. *More precisely*, $\sigma(a) + D(0,\epsilon) \subseteq \Lambda_{n,\epsilon}(a) \subseteq \sigma(a) + D(0,\epsilon + \eta_1)$
2. *For every* $\eta_2 > 0$, *there exists* $n_2 \in \mathbb{N}$ *such that* $d_H(\Lambda_{n,\epsilon_1}(a), \Lambda_{n,\epsilon_2}(a)) < |\epsilon_1 - \epsilon_2| + \eta_2$ *for all* $n \geq n_2$ *and all* $\epsilon_1, \epsilon_2 \leq \epsilon_0$.
3. *For every* $0 < \eta_3 < \epsilon_0$, *there exists* $n_3 \in \mathbb{N}$ *such that for all* $n \geq n_3$, *there exists a* $\delta(n) > 0$ *such that* $d_H(\Lambda_{n,\epsilon_1}(a), \Lambda_{n,\epsilon_2}(b)) < |\epsilon_1 - \epsilon_2| + \eta_3$ *for all* $\epsilon_1, \epsilon_2 \in [\frac{\eta_3}{4}, \epsilon_0]$ *and all* $b \in A$ *with* $\|a - b\| < \delta(n)$.

Remark 8.11 It is natural to ask similar questions about the condition spectrum, namely, what are the conditions for the continuity of the map $a \mapsto \sigma_\epsilon(a)$ or more generally the map $(\epsilon, a) \mapsto \sigma_\epsilon(a)$? As of now, no satisfactory answers are available to these questions. Some work on the level sets associated with the condition spectrum is reported in [22].

9 Concluding remarks

We shall conclude this survey by mentioning some results about the spectrum related sets without any elaborate discussion of those results. One idea is to look at some well known result about the spectrum and try to investigate what kind of analogues hold in case of these other sets. For example, there are large number of results in the literature on characterizing linear as well as nonlinear maps that preserve spectrum or numerical range or some numbers related to these sets such as spectral radius. Problems of this type are known as "Preserver Problems". A good account of such problems can be found in [24]. Some results about linear maps preserving pseudospectrum and condition spectrum are given in [19]. Analogues of the spectral mapping theorem for pseudospectrum and condition spectrum are discussed in [23] and [20] respectively. If A is a complex Banach algebra with unit 1 and p is an idempotent element in a, then pAp is also a Banach algebra with unit p. It is natural to investigate the relationship beteween the spectrum of a as an element of A and the spectrum of pap as an element of pAp and also similar questions about the other related sets. These are discussed in [15] and [16]. Similar studies in case of (n, ϵ)-pseudospectrum are reported in [8]. Suppose the elements a and b in a complex unital Banach algebra satisfy the following condition: $\Lambda_\epsilon(ax) = \Lambda_\epsilon(bx)$ for all $x \in A$. Then $a = b$ in certain situations. These are discussed in [14].

The problem of computing the spectrum of an element in a stable manner remains a challenging and interesting problem. Any such computation will involve computation of the functions $\gamma_n(a, z)$. The methods of computing $\gamma_n(T, z)$ effectively are known when T is a bounded operator on a separable Hilbert space. These are based on computing the singular values of the finite sections of T (see [11, 12] for details). Some methods of computing these sets using Banach algebra techniques are discussed in [21]. There are many issues involving computational complexity of this problem. A very interesting discussion on these issues can be found in [13].

In our discussion of stability of the spectrum and other sets in Section 8, we have considered the topology given by the norm on the Banach algebra under consideration and the Hausdorff metric on the subsets of the complex plane. When the algebra is of bounded operators on a Banach or Hilbert space and when one wants to discuss the approximation of the spectrum and related sets corresponding to an operator T by similar sets corresponding to its finite dimensional truncations T_n, one has to note that, in general, the sequence $\{T_n\}$ does not converge to T in norm. Hence other topologies like strong operator topology have to be considered. Also the notion of convergence of subsets of $\mathbb{C}$ needs to be changed. These ideas are persued by Arveson [1] and also by Böttcher, Silbermann [4] and their coauthors in different ways.

Arveson [1] has considered a sequence $\{H_n\}$ of finite dimensional subspaces of a Hilbert space H with some additional properties called *filtration* of H. He then defines the *degree* of an operator with respect to such a filtration. An operator of finite degree is a generalization of a band limited operator. It is then shown that if a self-adjoint operator T can be expressed as a sum of operators of finite degree, then some information about the essential spectrum of T can be obtained from the spectra of T_n.

In Böttcher and Silbermann [4], a sequence $\{T_n\}$ converging to T in the strong operator topology is called *stable* if there exists a natural number n_0 such that T_n is invertible for all $n \geq n_0$ and $\sup\{\|T_n^{-1}\|, n \geq n_0\}$ is finite. They consider an algebra F of bounded sequences $\{T_n\}$ and the ideal I of those sequences $\{T_n\}$ such that $\{\|T_n\|\}$ converges to 0. Then by a classical theorem of Kozak [4], a sequence $\{T_n\}$ is stable if and only if the coset $\{T_n\} + I$ is invertible in the quotient algebra F/I. Thus a question of stability becomes equivalent to a question of invertibility. This is called the *algebraization of stability* and this technique is used to get information about the spectra and pseudospectra of Toeplitz operators in [4].

In case of certain Banach algebras, their elements are naturally associated with some other objects. For example, in case of certain Banach algebras of Laurent and Toeplitz operarors, every such operarator is associated with a symbol which is a function. In such a situation, one would naturally want to know if any relation exists between such a symbol of a Toeplitz operator and and its spectrum and related sets. Some work dealing with this aspect about spectrum, pseudospectrum and condition spectrum can be found in [4] and [21]. Nothing much seems to be known about other related sets. In particular, we do not know any information regarding Ransford spectrum of a Toeplitz operator and its symbol.

Since we have confined our attention to the context of a Banach algebra, we could not deal with unbounded operators as such operators can not be members of any Banach algebra. Some times it so happens that even if an operator T is unbounded, the inverse of $\lambda I - T$ is a bounded operator for some values of λ. In case of such an operator, there is a natural way to define spectrum and ϵ-pseudospectrum (see [32]). In principle, this approach can be extended to (n, ϵ)-pseudospectrum. It is not clear whether and how the ideas of Ransford spectrum can be developed in this context.

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