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Gelfand–Mazur Theorems in normed algebras: A survey

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Abstract

The Gelfand–Mazur Theorem, a very basic theorem in the theory of Banach algebras states that: (Real version) Every real normed division algebra is isomorphic to the algebra of all real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$; (Complex version) Every complex normed division algebra is isometrically isomorphic to $\mathbb{C}$. This theorem has undergone a large number of generalizations. We present a survey of these generalizations and also discuss some closely related unsettled issues.

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1. The Gelfand–Mazur theorem

An *algebra* $\mathcal{A}$ over a field F is a vector space over F which is also a ring such that for all $x, y \in \mathcal{A}$ and for all $\lambda \in F$, $\lambda(xy) = x(\lambda y) = (\lambda x)y$ holds. We assume $\mathcal{A}$ to be associative and not necessarily having identity element. We shall take F to be either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, and accordingly call $\mathcal{A}$ to be a *real algebra* or a *complex algebra*. A *divisionalgebra* is an algebra with identity such that every non zero element is invertible. A *normed algebra* $(\mathcal{A}, \|.\|)$ is an algebra $\mathcal{A}$ together with a

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norm $\|.\|$ such that $(\mathcal{A}, \|.\|)$ is a normed linear space and the norm is submultiplicative, that is, $\|xy\| \leq \|x\|\|y\|$ for all x, y in $\mathcal{A}$. A *Banach algebra* is a normed algebra that is a Banach space. Banach algebras exhibit a fruitful interplay between Algebra and Analysis resulting into a rich theory of algebras in analysis [11,15,30,39,41]. The subject has a rich collection of examples from Function Theory, Harmonic Analysis and Linear Operator Theory in Banach and Hilbert Spaces. It has also provided a basic framework for the development of C^*-algebras and von Neumann algebras creating a foundation for the development of noncommutative mathematics of analysis like Noncommutative Probability and Noncommutative Geometry. The following fundamental theorem is a corner stone of Banach Algebras; and it compares in simplicity and beauty with the Liouville Theorem of Complex Analysis. We recall two popular versions of the theorem.

Theorem 1.1 (*Real Version*)**.** *Every real normed division algebra is isomorphic to the set of all real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$.*

Theorem 1.2 (*Complex Version*)**.** *Every complex normed division algebra is isometrically isomorphic to $\mathbb{C}$.*

The division algebra $\mathbb{H}$ of quaternions is the algebra consisting of elements of form $x = \alpha_0 1 + \alpha_1 i + \alpha_2 j + \alpha_3 k$ subject to the multiplication $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1$, 1 being the multiplicative identity. The theorem is a natural sequel to the classical Frobenius Theorem [17,20] that states that a real finite dimensional division algebra is isomorphic to $\mathbb{R}$, or $\mathbb{C}$ or $\mathbb{H}$; and it illustrates the power of methods of Analysis to study infinite dimensional algebras. This is also illustrated by the fact that in a Banach algebra, if an element x is invertible, then all y in an appropriate neighbourhood of x are also invertible. Immediately after the appearance of first papers in Banach algebras [37,49,50], Mazur [36] announced the theorem without proof. It is stated by some authors that Mazur's original submission contained a proof. But it was deleted from the final paper due to Editor's insistence on shortening the proof. A very elegant proof of the complex version, based on the Liouville Theorem for entire functions was given by Gelfand in his famous paper [22]. Mazur's original proof based incidentally on Liouville Theorem for harmonic functions became available much later in a book by Zelazko [53]. It can also be found in [34]. Thus chronologically the theorem deserves to be called the Mazur–Gelfand Theorem; but the term Gelfand–Mazur Theorem has become very popular and well established by now. Like some other fundamental theorems, Gelfand–Mazur Theorem and its avatars have also inspired elementary proofs thereof [14,21,29,33,40,41,45].

Let $\mathcal{A}$ be a complex normed algebra with identity 1. A proof due to Arens [2] of the complex version uses the Liouville Theorem. A major step in this proof is to prove that for $x \in \mathcal{A}$ the resolvent function $R_x(\lambda) := (\lambda 1 - x)^{-1}$, $\lambda \in \mathbb{C}$ is analytic wherever it is defined. A consequence of the theorem is the most fundamental result of Banach Algebras that for each $x \in \mathcal{A}$, the *spectrum* $sp(x) := \{\lambda \in \mathbb{C} : (\lambda 1 - x)$ is not invertible in $\mathcal{A}\}$ is non empty and compact. On the other hand, Gelfand's proof as well as the elementary proofs due to Kametani [31] and Rickart [40,41] establish first the non emptiness of spectra from which the theorem follows easily. (The elementary proof due to Kametani and Rickart is based on decomposing the polynomial $x^n - 1$ in terms of linear factors over complex scalars. This has inspired Almira [1] to give a proof of the Fundamental Theorem of Algebra using Gelfand–Mazur Theorem, though the proof uses the Frobenius Theorem.) Thus the Gelfand–Mazur Theorem is synonymous with the non emptiness of spectra. For commutative Banach algebras, the theorem paves the way for Gelfand Theory resulting into a passage to Function Theory. Indeed the maximal ideals, whose abundance in the semi simple case is ensured by Zorn's Lemma, result into normed division algebras by quotients; and then the Gelfand–Mazur Theorem produces scalar valued functions (Gelfand transforms) on the space of maximal ideals. If $\mathcal{A}$ is a real normed algebra with identity 1, then the above arguments need some modification. For $x \in \mathcal{A}$, the *spectrum* of x is defined by $sp(x) := \{s + it \in \mathbb{C} : (s1 - x)^2 + t^2$ is not invertible in $\mathcal{A}\}$. To show that this set is non empty and compact, one needs to use the theory of harmonic functions in stead of analytic functions. Also this set has an additional property, namely that it is symmetric with respect to the real axis, that is, $s + it \in sp(x)$ if and only if $s - it \in sp(x)$.(See [34] for details.)

This theorem has undergone a plethora of generalizations/extensions. The theorem has triggered a line of investigation in normed algebras (and in more general topological and other algebras) $\mathcal{A}$ searching for conditions on $\mathcal{A}$ that imply that $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. We call such a theorem a Gelfand–Mazur Theorem. Inspired by the aesthetic beauty of the theorem together with its fundamental role in Banach Algebras, we review such theorems highlighting some related unsolved questions. We give some background for each of these theorems, definitions required to understand the theorem, some comments about significance of the theorem and references. Proofs are not given. The search for Gelfand–Mazur Theorems in various classes of non-normed topological algebras is another related line of inquiry. We shall not discuss this aspect except at a remark at the end. Though every real number is a complex number with the result a complex Banach algebra is a real Banach algebra, the theory of real Banach algebras involves certain intricacies of conceptual nature and demand intrinsically real methods to work with. We refer to [34] for real Banach algebras.

Since we do not assume $\mathcal{A}$ to contain identity, we need to consider unitization $\mathcal{A}_e$ of $\mathcal{A}$ or to consider quasi-multiplication in $\mathcal{A}$. The algebra $\mathcal{A}_e$ is $\mathcal{A} \oplus F$ with pointwise linear operations and the multiplication $(x, \lambda)(y, \mu) = (xy + \lambda y + \mu x, \lambda\mu)$ so as to contain $\mathcal{A}$ as an ideal of codimension 1 via the map $x \in \mathcal{A} \to (x, 0) \in \mathcal{A}_e$. The quasi-multiplication in $\mathcal{A}$ is $xoy = x + y - xy$. An element $x \in \mathcal{A}$ is quasi-regular having quasi-inverse $x_{-1} := y$ if $xoy = yox = 0$; and this happens if and only if $(x, 1)$ is invertible in $\mathcal{A}_e$. Then for $x \in \mathcal{A}$, $sp_{\mathcal{A}}(x) := \{\lambda \in \mathbb{C} : -\lambda^{-1}x$ is not quasi regular$\}$ = $sp_{\mathcal{A}_e}(x, o)$. For details, we refer to [11,35,41].

2. Spectral algebras

An idea that evolved over a time with contributions from Kaplansky, Michael, Yood and Palmer, and very well exposed in Palmer [38,39] is that much of the success of Banach algebra theory is a consequence of the fact that invertible elements (quasi-regular elements, in case of absence of identity) form an open set. This is incorporated in spectral algebra [38,39]. A *spectral seminorm* p on an algebra $\mathcal{A}$ is a submultiplicative seminorm p on $\mathcal{A}$ such that the set of quasi-regular elements is open with respect to p. A *semi* *spectral algebra* (respectively a *spectral algebra*) is an algebra with a spectral seminorm (respectively spectral norm). An elementary argument, due to Rickart as modified by Palmer [39] shows that in an algebra $\mathcal{A}$ with a submultiplicative seminorm p, the spectrum $sp(x)$ is non empty for all $x \in \mathcal{A}$, and $r_p(x) := \lim_{n\to\infty} p(x^n)^{1/n} \le \sup\{|\lambda| : \lambda \in sp(x)\} =: r(x)$; and p is spectral iff the celebrated spectral radius formula $r_p(x) = r(x)$ hold for all x[5,39]. This quickly leads to the following [39].

Theorem 2.1. *Let $\mathcal{A}$ be a real division algebra on which a non trivial submultiplicative seminorm is defined. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

The complex case of Theorem 2.1 is discussed in Theorem 2.2.3 in [39]; and the real case follows using Frobenius Theorem. A real division algebra $\mathcal{A}$ is $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ if $\mathcal{A}$ is a homomorphic image of a normed algebra or if $\mathcal{A}$ admits a non trivial homomorphism into a normed algebra. This also holds if $\mathcal{A}x = x\mathcal{A} = \mathcal{A}$ for all x. Notice that the existence of a submultiplicative seminorm on an algebra $\mathcal{A}$ is a purely algebraic condition equivalent to the existence of a convex balanced absorbing multiplicative subsemigroup S [11]; and this is a non trivial matter. The Arens algebra $L^{\omega}[0, 1] := \cap_{1\le p<\infty} L^p[0, 1]$ with pointwise multiplication admits no non trivial submultiplicative seminorm, as it admits no nonzero multiplicative linear functional [52]. Further $\mathcal{A}$ is semi spectral iff there exists S as above contained in quasi-regular elements [39]. The fact that being a spectral algebra is equivalent to the validity of the spectral radius formula is related with a question [15] whose answer awaits a solution: Characterize an algebra that is a Banach algebra under some norm. It would be worth while looking to spectral algebra analogues of other Gelfand–Mazur Theorems discussed below.

3. Kaplansky's theorem

The presence of a metric structure compatible with the algebraic structure on a normed algebra helps topologizing algebraic concepts. This is best illustrated by the topological divisors of zero (TDZ). Following Definition 2.12 in [11], given a normed algebra $\mathcal{A}$, an element x in $\mathcal{A}$ is a *joint topological divisor of zero* if there is a sequence $\{y_n\}$ in $\mathcal{A}$ such that $\|y_n\| = 1$ for each n and $xy_n \to 0$, $y_n x \to 0$. Kaplansky [31] has shown that in a normed algebra, being a division algebra is equivalent to the algebra not admitting a non zero TDZ. A simple proof has been given [14].

Theorem 3.1. *Let $\mathcal{A}$ be a real normed algebra with no non zero joint TDZ. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

The conclusion holds if $\mathcal{A}$ has no non zero left (respectively right) TDZ. The paper [14] also contains a non associative alternative algebra version.

4. Edwards theorem

The following result due to Edwards [18] gives a metric condition for Gelfand–Mazur phenomenon.

Theorem 4.1. *Let $\mathcal{A}$ be a real unital Banach algebra such that $\|x^{-1}\| \leq \|x\|^{-1}$ for all invertible elements x in $\mathcal{A}$. Then $\mathcal{A}$ is a division algebra and is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

Thus a Banach algebra with identity which is also an absolute valued algebra (i.e. the norm satisfies $\|xy\| = \|x\|\|y\|$ for all $x, y \in \mathcal{A}$) is a division algebra. Now suppose $\mathcal{A}$ is a spectral normed algebra. Then $\mathcal{A}$ is inverse closed in its Banach algebra completion $\tilde{\mathcal{A}}$ and $\mathcal{A}^{-1} = (\tilde{\mathcal{A}})^{-1} \cap \mathcal{A}$. A simple argument with completion gives a spectral algebra version of Edward's Theorem; viz.; Let $\mathcal{A}$ be a real spectral algebra with identity such that for every invertible x in $\mathcal{A}$, $\|x^{-1}\| \leq \|x\|^{-1}$ holds. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. We do not know a non unital version of Edwards Theorem. The following results (Corollary 3.2.2 and Corollary 3.2.3 in [3]) closely related with Edward's Theorem are due to Aupetit.

Theorem 4.2. *Let $\mathcal{A}$ be a real Banach algebra with identity. Let U be a non empty open subset of $\mathcal{A}$ consisting of invertible elements.*

(a) *Suppose that for each $x \in U$, $r(x)r(x^{-1}) \leq 1$. Then $\mathcal{A}/rad\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ or the algebra $M_2(\mathbb{R})$ of 2×2 real matrices.*

(b) *Suppose that for each $x \in U$, $\|x\|\|x^{-1}\| \leq 1$. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

(c) *Suppose $\mathcal{A}$ is a complex algebra. Then in each of above* (a) *and* (b), *$\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

5. Integral domains and Gelfand–Mazur phenomena

The Gelfand–Mazur–Kaplansky Theorem suggests: When is a Banach algebra that is an integral domain a division algebra? Notice that the disc algebra $A(D)$ consisting of functions continuous on the closed unit disc D and holomorphic in its interior is a Banach algebra with supnorm $\|f\|_\infty := \sup\{|f(z)| : z \in D\}$ and is an integral domain. The sup norm is a *uniform norm* $\|.\|$ in the sense that it satisfies $\|f^2\| = \|f\|^2$ for all $f \in A(D)$. In fact the disc algebra admits a continuum of (not necessarily complete) uniform norms $\|f\|_{r,\infty} = \sup\{|f(z)| : |z| \leq r\}$ for $0 < r < 1$. On the other hand, one says that a Banach algebra $\mathcal{A}$ has *Unique Uniform Norm Property (UUNP)* if $\mathcal{A}$ admits exactly one uniform norm (we emphasize, not necessarily complete). Notice that the presence of a uniform norm on a complex Banach algebra ensures that the algebra is commutative. On a commutative Banach algebra $\mathcal{A}$, the spectral radius $r(.)$ is a uniform seminorm, which is a norm iff $\mathcal{A}$ is semi simple. The property UUNP turns out to be closely related with Silov regularity [8]. The complex case of the following has been proved in [8]. The following general case is due to Jarosz and Kulkarni [28].

Theorem 5.1. *A real Banach algebra $\mathcal{A}$ with a unique uniform norm has no nonzero zero divisors if and only if $\mathcal{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

An algebra $\mathcal{A}$ with identity 1 is *locally finite* if for each $h \in \mathcal{A}$, the smallest sub algebra (not necessarily closed) containing 1 and h is finite dimensional. The following [9] has been proved in the commutative complex case by Srinivasan [24,46].

Theorem 5.2. *Let $\mathcal{A}$ be a real Banach algebra with identity. Suppose $\mathcal{A}$ has no nonzero zero divisors. If $\mathcal{A}$ is locally finite, then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

The proof in [9] is based on a real algebra analogue of a result of Srinivasan; viz.,if $\mathcal{A}$ is a real Banach algebra that is an integral domain, then (a) for any non zero $h \in \mathcal{A}$, h is not a TDZ iff the principal ideal (h) generated by h is closed; and (b) if for all non zero $h \in \mathcal{A}$, (h) is closed, then $\mathcal{A}$ is a division algebra. It follows that a finite dimensional Banach algebra with identity having no non-zero zero divisors is a division algebra. In above theorem, in case of algebra without identity, one can obviously modify local finiteness, and ask whether an analogue of above theorem holds in this case.

A commutative ring is *Noetherian* if every properly ascending chain of ideals terminates; equivalently, if every non empty set of ideals contain a maximal element. This happens iff every ideal is finitely generated. Sinclair and Tullo [44] in the commutative case and Aupetit [3] in the real case have shown that a real Noetherian Banach algebra is finite dimensional. This led Srinivasan [46] to show that additionally if $\mathcal{A}$ is also an integral domain, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$. The following [9] is at the desired level of generality.

Theorem 5.3. *Let $\mathcal{A}$ be a real Banach algebra with identity. Let $\mathcal{A}$ be Noetherian. Then $\mathcal{A}$ is finite dimensional. Further if $\mathcal{A}$ is also an integral domain, then $\mathcal{A}$ is $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

Thus a real Banach algebra with identity which is a principal ideal domain (PID) is a division algebra. The polynomial algebra is a classic counterexample in the absence of completeness. Presumably the above theorem also holds for a spectral algebra. What happens in the case of unique factorization domain (UFD)? The following partial answer is a real analogue of a result from [47].

Theorem 5.4. *Let $\mathcal{A}$ be a real Banach algebra with identity. Let $\mathcal{A}$ be a UFD. If for each prime element p, (p) is closed, then $\mathcal{A}$ is a division algebra.*

A *ring of valuation* is a commutative ring with identity which is also an integral domain in which the set of all principal ideals form a chain. The following is due to Esterle [19]. Its real analogue is awaited.

Theorem 5.5. *Let $\mathcal{A}$ be a complex commutative Banach algebra which is a ring of valuation. Then $\mathcal{A}$ isomorphic to $\mathbb{C}$.*

A *-*algebra* is an algebra with an involution * which is a conjugate linear anti automorphism of period 2. A *Banach* *-*algebra* is a Banach algebra which is a *-algebra such that $\|x^*\| = \|x\|$ for all $x \in \mathcal{A}$. A (complex) C^*-*algebra* is a complex Banach *-algebra $(\mathcal{A}, \|.\|)$ such that $\|x^*x\| = \|x\|^2$ for all x. A *real* C^*-*algebra* [23] is a real Banach *-algebra $(\mathcal{A}, \|.\|)$ such that for all $x \in \mathcal{A}$, $\|x^*x\| = \|x\|^2$ and $1 + x^*x$ is invertible in the unitization $\mathcal{A}_e$ of $\mathcal{A}$. An element x in a C^*-algebra is said to be positive, if $x = u^*u$ for some $u \in \mathcal{A}$. It is known that every positive element has a unique positive square root [11]. The positive square root of x^*x is denoted by $|x|$. The following [9] illustrates that in some of Gelfand–Mazur Theorems, the stated assumptions can be weakened in the presence of involutive structure. Let $Sym(\mathcal{A})$ consist of all those $h \in \mathcal{A}$ that are symmetric, that is, $h = h^*$.

Theorem 5.6. (a) *Let $\mathcal{A}$ be a real C^*-algebra. Assume that at least one of the following holds.*

(i) *$\mathcal{A}$ has identity, $|h| = +h$ or $-h$ for all $h \in Sym(\mathcal{A})$*

(ii) *$Sym(\mathcal{A})$ has no nonzero zero divisors.*

Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

(b) *Let $\mathcal{A}$ be a complex C^*-algebra satisfying at least one of above* (i) *or* (ii) *or the following.*

(iii) *$\mathcal{A}$ has identity and for each $h \in Sym(\mathcal{A})$, $sp(e^{ih})$ is convex.*

Then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

A Banach *-algebra is said to be*Hermitian* if for every $h \in Sym(\mathcal{A})$, $Sp(h) \subset \mathbb{R}$. It would be worth while searching for a analogue of above assuming $\mathcal{A}$ to be H Hermitian rather than a C^*-algebra. Note that a C^*-algebra is Hermitian [11].

6. Theorems of Ingelstam and Zalar

Ingelstam [26,27] proved that if $\mathcal{A}$ is a real algebra with identity 1 such that $\mathcal{A}$ is also a real Hilbert space with $\|1\| = 1$ and $\|xy\| \leq \|x\|\|y\|$ for all x, y in $\mathcal{A}$, then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. Ingelstam's proof is based on the geometric notion of vertex property. Progressively simpler proofs have been evolved simultaneously with the strengthening of the theorem. The proofs by Smiley [45] and Froehlich [21] use Gelfand Theory coupled with geometry of Hilbert space. That the existence of identity cannot be omitted is exhibited by the sequence algebra ℓ^2. The algebra $\mathbb{C}^2$ with point wise multiplication and Euclidean norm shows that the condition $\|1\| = 1$ also cannot be omitted. However completeness of $\mathcal{A}$ can be dispensed with and the norm inequality can be replaced by the weaker square inequality. This is the content in the following result due to Zalar [51]. His proof is based on geometric arguments ultimately appealing to TDZ and the Gelfand–Mazur Theorem.

Theorem 6.1. (a) *Let $\mathcal{A}$ be a real algebra with identity 1. Suppose $\mathcal{A}$ is also a real inner product space such that $\|x^2\| \leq \|x\|^2$ for all x and $\|1\| = 1$. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

(b) *Let $\mathcal{A}$ be a real algebra which is a real inner product space such that $\|x^2\| = \|x\|^2$ for all x. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

A very simple and elementary proof of above theorem is given in [33]. It only uses simple geometry of inner product space together with elementary algebra, and is accessible to under graduates also. Zalar [51] has also discussed an alternative algebra analogue of the above theorem.

7. An involutive algebra analogue of Zalar's Theorem

The apparent similarity between the square property $\|x^2\| = \|x\|^2, x \in \mathcal{A}$ of the complete norm in a uniform Banach algebra and the C^*-property $\|x^*x\| = \|x\|^2, x \in \mathcal{A}$ of the C^*-norm in a C^*-algebra suggested a structural analogy between uniform Banach algebras and C^*-algebras [6,7]. In the light of this, Zalar's Theorem discussed above inspired the following in [9]. A norm on $\mathcal{A}$ is said to be *Pythagorean* if it is induced by an inner product. Such a norm satisfies the following identity, known as the parallelogram identity.

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } \quad x, y \in \mathcal{A}.$$

Theorem 7.1. *Let $\|.\|$ be a Pythagorean norm on a real *-algebra $\mathcal{A}$.*

(a) *Assume that the norm $\|.\|$ satisfies at least one of the following.*

(i)(1) *$\mathcal{A}$ has identity element 1, $\|1\| = 1$, $\|x^*x\| \leq \|x\|^2$ for all $x \in \mathcal{A}$.*

(2) *$x^*x = 0$ implies $x = 0$*

(ii) *$\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$.*

Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

(b) *Suppose $\mathcal{A}$ is a complex *-algebra. Assume at least one of the following.*

(i) *$\mathcal{A}$ has identity element 1, $\|1\| = 1$, $\|x^*x\| \leq \|x\|^2$ for all $x \in \mathcal{A}$.*

(ii) *$\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$.*

Then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

This is derived from the following [9] that seems to be of some independent interest.

Theorem 7.2. *Let $\mathcal{A}$ be a real *-algebra with identity. Assume that $x^*x = 0$ implies $x = 0$. Let $\|.\|$ be a norm on $Sym(\mathcal{A})$ such that for all x, y in $Sym(\mathcal{A})$ with $xy = yx$, it holds that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. Assume at least one of the following.*

(i) *$\|1\| = 1$, $\|x^2\| \leq \|x\|^2$ for all $x \in \mathcal{A}$.*

(ii) *$\|x^2\| = \|x\|^2$ for all $x \in Sym(\mathcal{A})$.*

Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.

Here are some interesting issues arising from above theorems. Let $u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $u \neq 0$, $u^2 = 0$ and the unital *-algebra $\mathcal{A} := \{\alpha + \beta u : \alpha, \beta \in \mathbb{R}\}$ with involution $(\alpha + \beta u)^* = \alpha - \beta u$ exhibits that the condition $x^*x = 0$ implies $x = 0$ cannot be omitted. The algebra $\mathbb{R} \times \mathbb{R}$ with pointwise multiplication and the Euclidean norm shows that the condition $\|1\| = 1$ is essential. Theorem 5.1 suggests this: Let $\mathcal{A}$ be a *-semi simple Banach *-algebra admitting a unique (not necessarily complete) C^*-norm. Suppose $\mathcal{A}$ has no nonzero zero divisors. Is $\mathcal{A}$ a division algebra? The well known norm characterizations of C^*-algebras [11] by Glimm and Kadision and by Vowden suggest this : In above theorems, can the conditions $\|x^*x\| \leq \|x\|^2$ and $\|x^*x\| = \|x\|^2$ be replaced by the respective conditions $\|x^*x\| \leq \|x^*\|\|x\|$ and $\|x^*x\| = \|x^*\|\|x\|$? Here is another elementary question: Let p be a seminorm on an algebra $\mathcal{A}$ with identity 1 such that $p(1) = 1$. Let either (a) $p(x^2) \leq p(x)^2$ for all x, or (b) $\mathcal{A}$ is a *-algebra and $p(x^*x) \leq p(x)^2$ for all x. If p is Pythagorean, is $p(xy) \leq p(x)p(y)$ for all x, y? The answer is affirmative if $\mathcal{A}$ is commutative [9]. This question is related with two interesting results. One is due to Sebestyn [43] stating that a seminorm p on a *-algebra $\mathcal{A}$ satisfying the C^*-property $p(x^*x) = p(x)^2 (x \in \mathcal{A})$ is necessarily submultiplicative; the other one is a square property analogue due to Dedania [16] stating that a seminorm on any algebra $\mathcal{A}$ satisfying $p(x^2) = p(x)^2 (x \in \mathcal{A})$ is submultiplicative.

8. Miscellaneous results

(a) **Spectral Convexity**: Let $(\mathcal{A}, \|.\|)$ be a complex Banach algebra with identity. The *numerical range* $V(x)$ [10,12] of an element $x \in \mathcal{A}$ is a geometric entity defined as $V(x) = \{f(x) : f \in \mathcal{A}', \|f\| = f(1) = 1\}$. In case of some special elements (for example normal elements in a C^*-*algebra*) the numerical range provides a convex approximation of the spectrum of x as $V(x) = \overline{co}sp(x)$, the closure of the convex hull of $sp(x)$. Here $\mathcal{A}'$ is the dual of $\mathcal{A}$. This has motivated the following [4]. For a Banach algebra $\mathcal{A}$, its radical $rad\mathcal{A}$ is the intersection of all maximal modular left ideals of $\mathcal{A}$.

Theorem 8.1. *Let $\mathcal{A}$ be a complex Banach algebra with identity 1. If $sp(x)$ is convex for all $x \in \mathcal{A}$, then $\mathcal{A}/rad\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

The proof involves showing $sp(x)$ to be a singleton for all x leading to an appeal to the Gelfand–Mazur Theorem. If $\mathcal{A}$ does not have an identity, then one shows that $rad\mathcal{A} = \mathcal{A}$.

There does not exist any good notion of numerical range of an element of a real Banach algebra and there does not exist any well developed theory about it. Also the spectrum of an element λ in $\mathbb{C}$, regarded as a real algebra, is $\{\lambda, \bar{\lambda}\}$, which is in general not a convex set. Hence it is not clear what should be the natural analogue of the above theorem for real algebra. It would be of interest to know whether the following is true: Let $\mathcal{A}$ be a real Banach algebra with identity 1. If $sp(x)$ is convex for all $x \in \mathcal{A}$, then $\mathcal{A}/rad\mathcal{A}$ is isomorphic to $\mathbb{R}$.

On the other hand, the above result may hold for complex spectral algebra also. Theorem 8.1 can be modified in the presence of involutive structure. In a Banach *-algebra $\mathcal{A}$, it suffices to assume that $sp(x)$ is convex for all normal elements, though the convexity of spectra for all self adjoint elements does not suffice. For Hermitian Banach *-algebras, it is sufficient to assume convexity of spectra for all unitary elements.

Analogous results have also been discussed in [3]. Here is a typical result. Let $\mathcal{A}$ be a complex Banach algebra. If there exists a non empty open subset U in $\mathcal{A}$ such that $sp(x)$ is a singleton for all $x \in U$, then $\mathcal{A}/rad\mathcal{A}$ is isomorphic to $\mathbb{C}$. More generally Theorem 3.2.1 in [3] states that if $\mathcal{A}$ is a real Banach algebra containing a non empty open set U such that $sp(x)$ is finite for all $x \in U$, then $\mathcal{A}/rad\mathcal{A}$ is finite dimensional. A generalization of Gelfand–Mazur Theorem developed in [32] states that a unital semi simple complex Banach algebra $\mathcal{A}$ which has only trivial idempotents and in which $sp(x)$ is countable for each x is isomorphic to $\mathbb{C}$.

(b) **Numerical range characterization**

Theorem 8.2 (*[12]*)**.** *Let $\mathcal{A}$ be a complex Banach algebra with identity.*

(a) *If for all $x \in \mathcal{A}$, $V(x^2) \subseteq \{\lambda^2 : \lambda \in V(x)\}$, then $\mathcal{A}$ is homeomorphically isomorphic to a uniform Banach algebra.*

(b) *If for all $x \in \mathcal{A}$, $V(x^2) = \{\lambda^2 : \lambda \in V(x)\}$, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

(c) O^***-algebra version of Gelfand–Mazur Theorem**

Let D be a dense subspace of a complex Hilbert space $\mathcal{H}$. Let $\mathcal{L}^+(D)$ consists of all linear operators $T : D \to \mathcal{H}$, not necessarily bounded, such that $T(D) \subseteq D$, $D \subseteq Dom(T^*)$ and $T^*D \subseteq D$. Then $\mathcal{L}^+(D)$ is a *-algebra with usual vector wise linear operations, composition as multiplication and the involution $T \to T^+ := T^*|_D$. An *O^*-algebra* is a *-sub algebra of $\mathcal{L}^+(D)$. These unbounded operator algebras and their representation theory have been conveniently summarized in [42]. These are far away from being normable (even being seminormable). Still surprisingly the following [42] holds.

Theorem 8.3. *Let $\mathcal{A}$ be a complex O^*-algebra which is a division algebra. Then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Under the stated assumption, for any $T = T^+ \in \mathcal{A}$, the closure $\bar{T}$ turns out to be self adjoint (a priori unbounded) to which the Spectral Theorem applies. Thus it is the operator theoretic involutive structure that works. For an unbounded operator algebra that is not involutive, the theorem may not hold. However, we have no counterexample. Note that there are non trivial topological fields [36,48]; distribution theory also provides non trivial convolution algebras of distributions that are fields.

(d) **Alternative algebra analogues**

Let $\mathcal{A}$ be a non-associative algebra. It is *alternative* if for all $x, y \in \mathcal{A}$, $x^2y = x(xy)$ and $yx^2 = (yx)x$. An alternative algebra analogue of the classical Frobenius Theorem [17] states that a finite dimensional real non-associative alternative algebra that is a division algebra is isomorphic to the reals $\mathbb{R}$ or the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$ or the octonions $\mathbb{O}$. In fact, these are the only real finite dimensional division algebras [13]. In the frame work of real non-associative normed algebras, this could inspire a search for non-associative analogues of various Gelfand–Mazur Theorems. The following [14] is an analogue of Kaplansky's Theorem.

Theorem 8.4. *Let $\mathcal{A}$ be a real non associative alternative normed algebra with no non-zero joint topological divisors of zero. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ or $\mathbb{O}$.*

The following from [51] gives alternative algebra analogues of Theorem 6.1.

Theorem 8.5. (a) *Let $\mathcal{A}$ be an alternative real algebra with identity 1 which is also an inner product space. Suppose $\|x^2\| \leq \|x\|^2$ holds for all $x \in \mathcal{A}$ and $\|1\| = 1$. Then $\mathcal{A}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ or $\mathbb{O}$.*

(b) *Let $\mathcal{A}$ be a not necessarily associative complex algebra with identity 1 which is also an inner product space. Suppose $\|x^2\| \leq \|x\|^2$ holds for all $x, y \in \mathcal{A}$ and $\|1\| = 1$. The $\mathcal{A}$ is isomorphic to $\mathbb{C}$ and is necessarily associative.*

Non-associative algebra analogues of Theorems 6.1(b) and 7.1 as well as of other Gelfand–Mazur Theorems discussed above are not yet known and should be of some interest.

(e) **Remarks**

Discussion of Gelfand–Mazur Theorems in more general topological algebras has been of some interest. Theorem 1.1 has been generalized for various classes of topological algebras. Arens [2] showed that the theorem holds in a real algebra that is a locally convex space in which the multiplication and inversion are continuous. In fact a topological division algebra having continuous inversion and a total set of continuous linear functionals is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$. It also holds in a separable locally convex complete metric algebra. On the other hand, Zelazko [53] showed that the theorem holds in a complete locally bounded (not necessarily locally convex) algebra. However the following case does not seem to have been considered. An *s-normed algebra* $(\mathcal{A}, \|.\|)$ is a linear associative algebra that is a normed linear space such that $\|xy\| \leq \|x^2\|^{1/2}\|y^2\|^{1/2}$ for all x, y. This class of algebras arises in the study of orthogonal bases [25]. Does the Gelfand–Mazur Theorem hold in an s-normed algebra? One may also look for real algebra analogues of Theorems 5.1, 5.5, 8.1, 8.2, as well as analogues of Theorems 5.2 and 5.3 for algebras without identity.

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