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COMPLETENESS AND INVERTIBILITY

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Abstract. We show that two very important concepts in Functional Analysis, namely the completeness of a normed linear space and invertibility of a bounded linear map are related to each other. This gives a possibly new characterization of completeness.

1. Introduction

Answers to many important questions in Functional Analysis depend upon knowing whether a certain normed linear space is complete or/and whether a certain bounded linear map has a bounded linear inverse. Usually students do not think that these two important ideas in Functional Analysis, namely completeness and invertibility, have anything to do with each other. In this note, we try to draw the attention of students to connections between these ideas.

The following well known theorem is given in many textbooks of Functional Analysis. (See for example, [1].)

Theorem 1.1. *Let T be a bounded(continuous) linear map from a Banach space X to a normed linear space Y. Then the following are equivalent:*
1. T has a bounded inverse.
2. T is bounded below and the range of T is dense in Y.

It is natural to ask what happens if the hypothesis of completeness of X is dropped. Somehow, this question is not discussed in the textbooks. It is obvious that (1) would still imply (2) even without completeness. But the converse is false and it is easy to construct a counterexample. We give such an example. Further, it is interesting to note that (2) is equivalent to the following even without the completeness of X.
3. The transpose T' of T has a bounded inverse.
Even more interesting is the fact that the completeness of X is equivalent to the invertibility of every bounded linear map satisfying (2).

2. preliminaries

We recall a few standard notations, definitions and results that are used in the next section. For normed linear spaces X, Y, we denote by $BL(X,Y)$ the set of

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all bounded linear operators from X to Y. For an operator $T \in BL(X,Y)$, $N(T)$ denotes the null space of T and $R(T)$ denotes the range of T. Thus

$$N(T) = \{x \in X : T(x) = 0\} \qquad \text{and} \qquad R(T) = \{T(x) : x \in X\}.$$

Further, T is said to be *bounded below* if there exists $\alpha > 0$ such that $\|T(x)\| \geq \alpha\|x\|$ for all $x \in X$ and T is said to be *invertible* if there exists $S \in BL(Y,X)$ such that $ST = I_X$, the identity map on X, and $TS = I_Y$, the identity map on Y. The *dual space* X' of X, is the set of all bounded linear functionals on X, that is, $X' = BL(X, \mathbb{K})$, where $\mathbb{K}$ is the underlying field of real or complex numbers. For a subset $A \subseteq X$, the *annihilator* A^0 is the set of all continuous linear functionals that vanish on A, that is, $A^0 := \{\phi \in X', \phi(a) = 0 \text{ for all } a \in A\}$. If A is a subspace of X, then it follows by the Hahn-Banach Theorem, that A is dense in X, if and only if $A^0 = \{0\}$. The *transpose* T' of $T \in BL(X,Y)$ is the operator in $BL(Y', X')$ defined by
$(T'\psi)(x) := \psi(T(x))$ for all $x \in X$ and $\psi \in Y'$. All the other notations (including the notations for sequence spaces c_{00}, ℓ^1 etc.) are as in [1] and [2]. We shall make use of the following well known results:
1. $(R(T))^0 = N(T')$.
2. Every normed linear space X can be viewed as a dense subspace of a Banach space which we shall denote by X_c. (More precisely, there is a linear isometry of X onto a dense subspace of X_c.) The Banach space X_c is called the *completion* of X. These results can be found in any book on Functional Analysis, for example [1] and [2].

3. Notes

We begin with an example.

Example 3.1. Let $X := (c_{00}, \|.\|_1)$, $Y := \ell^1$ and $T : X \to Y$ be given by $T(x) = x$ for $x \in X$. Clearly, T is bounded below, range of T is dense in Y, but T is not onto and hence not invertible. More generally, we can consider the inclusion map from a proper dense subspace of a normed linear space.

Remark 3.2. Note that in the above example, though T is not invertible, its transpose T' is invertible. In fact, both the dual spaces X' of X and Y' of Y can be identified with ℓ^∞ in the usual way (See [2] for details.) and with respect to this identification T' becomes the identity operator on ℓ^∞.

This leads to some natural observations. First we consider some elementary results. The following elementary result is given as an Exercise in some books.

Lemma 3.3. *Let T be a bounded(continuous) linear map from a normed linear space X to a normed linear space Y. Then $R(T)$ is dense in Y if and only if T' is injective.*

Proof. Recall that $R(T)$ is dense in Y if and only if $\{0\} = (R(T))^0 = N(T')$ if and only if T' is injective. $\square$

Lemma 3.4. *Let T be a bounded(continuous) linear map from a normed linear space X to a normed linear space Y. Then the following statements are equvivalent:*
1. T has a bounded inverse from $R(T)$ to X.
2. T is bounded below.
3. T' is onto.

Proof. (1) implies (2): This is easy. Suppose $S : R(T) \to X$ is a bounded inverse of T. Then for each $x \in X$,
$\|x\| = \|ST(x)\| \le \|S\|\|T(x)\|$, that is, $\|T(x)\| \ge \frac{1}{\|S\|}\|x\|$.

(2) implies (1) and (3): Since T is bounded below, there exists $\alpha > 0$ such that $\|T(x)\| \ge \alpha\|x\|$ for all $x \in X$. In particular, T is injective. Hence we can define a map $S : R(T) \to X$ by $S(y) = x$ for $y = T(x) \in R(T)$. This is well defined since T is injective. It is easy to see that S is linear. Also
$\|S(y)\| = \|x\| \le \frac{1}{\alpha}\|T(x)\| = \frac{1}{\alpha}\|y\|$.
Hence S is bounded. This proves (1).

Next let $\phi \in X'$ and $y \in R(T)$. There exists unique $x \in X$ such that $y = T(x)$. Define ψ by $\psi(y) := \psi(T(x)) = \phi(x)$. This defines ψ as a linear functional on $R(T)$. Further,
$|\psi(y)| = |\psi(T(x))| = |\phi(x)| \le \|\phi\|\|x\| \le \|\phi\|\frac{1}{\alpha}\|T(x)\| = \|\phi\|\frac{1}{\alpha}\|y\|$.
This shows that ψ is bounded on $R(T)$ and hence has a bounded (norm preserving) extension to Y by the Hahn-Banach Theorem. We denote this extension also by the same symbol ψ. Thus $\psi \in Y'$ and $\phi = T'(\psi)$. This shows that T' is onto.

(3) implies (2): Let $x \in X$. By the Hahn-Banach Theorem, there exists $\phi \in X'$ such that $\phi(x) = \|x\|$ and $\|\phi\| = 1$. Further, since T' is onto, there exists $\psi \in Y'$ such that $\phi = T'(\psi)$. Now
$\|x\| = \phi(x) = T'(\psi)(x) = \psi(T(x)) \le \|\psi\|\|T(x)\|$, that is, $\|T(x)\| \ge \frac{1}{\|\psi\|}\|x\|$
This shows that T is bounded below. □

We now give the main theorem.

Theorem 3.5. *Let T be a bounded(continuous) linear map from a normed linear space X to a normed linear space Y. Consider the following statements:*
1. T has a bounded inverse.
2. T is bounded below and the range of T is dense in Y.
3. T' is invertible.

Then (2) and (3) are equivalent and each is implied by (1). If, in addition, X is a Banach space, then all the three statements are equivalent.

Proof. (1) implies (2): Obvious. Since T has a bounded inverse, $R(T) = Y$. Also T is bounded below by Lemma 3.4.

(2) if and only if (3): By Lemma 3.3, $R(T)$ is dense in Y, if and only if T' is injective. Further, by Lemma 3.4, T is bounded below, if and only if, T' is onto. Thus (2) is equivalent to the following:

$T' : Y' \to X'$ is a bijection.
Hence T' is invertible by the Closed Graph Theorem as X', Y' are Banach spaces.

Finally, if X is a Banach space, then (2) implies (1) by Theorem 1.1 and hence all the three statements are equivalent. □

Remark 3.6. We may further note that completeness of X is , in fact, equivalent to the invertiblity of every bounded linear map satisfying (2). In other words, a normed linear space X is a Banach space if and only if every bounded linear map T from X to any normed linear space Y such that T is bounded below and the range of T is dense in Y, is invertible. The only if part is already proved above. To prove the if part, consider $Y = X_c$, the completion of X. Then there is a linear isometry T of X onto a dense subspace Y_0 of Y. (See [2] for details.) Obviously, this T is bounded below and $R(T) = Y_0$ is dense in Y. Hence by the hypothesis, T is invertible and, in particular, onto. Thus X is linearly isometric to Y and hence complete.

Remark 3.7. It is known that the invertibility of an operator is closely related to its spectrum. Let X be a complex normed linear space and $T \in BL(X, X)$. Recall that the *spectrum* $\sigma(T)$ of T is the set of all complex numbers λ such that $\lambda I - T$ is not invertible. Applying Theorem 3.5 to $\lambda I - T$, we obtain the known result that $\sigma(T') \subseteq \sigma(T)$ and the equality holds if X is a Banach space. (See [2]) (The inclusion can be strict if X is not a Banach space. See the next example.) A natural question is whether the converse holds. In other words, can the completeness be also characterized in terms of spectra as follows: A complex normed linear space X is complete if and only if $\sigma(T') = \sigma(T)$ for all $T \in BL(X, X)$? Another formulation of the same question is as follows: Given an incomplete normed linear space X, does there exist $T \in BL(X, X)$ such that T is bounded below, its range is dense in X and T is not invertible (that is, not onto)? Note that the above examples and remarks do not answer this question as the spaces X and Y considered there are different.

Example 3.8. This example shows that the inclusion $\sigma(T') \subseteq \sigma(T)$ can be strict if X is not complete.

Let $X := (c_{00}, \|.\|_2)$, and $T : X \to X$ be the right shift operator given by $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ for $x := (x_1, x_2, \ldots) \in X$. Then the dual space X' can be identified with ℓ^2 and the transpose T' of T can be identified with the left shift operator. (See [2] for details.) Then it can be shown that
$\sigma(T') = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disc. On the other hand, it is easy to see that the equation $(\lambda I - T)x = e_1 = (1, 0, 0, \ldots)$ has no solution $x \in X$ for any complex number λ. In other words, $\lambda I - T$ is not onto. Thus $\sigma(T) = \mathbb{C}$.

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